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Modular Invariants for the Affine Algebra $C_2^{(1)}$

by

Phoebe Jane Elliot

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of **Master of Science**.

in

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To my father for inspiring my love of mathematics.
To my mother for her loving support of all my endeavours.

Abstract

This thesis concerns the classification of the modular invariant matrices associated to the affine algebra $C_2^{(1)}$ at level k. We provide a foundation for this classification and give a conjecture for the full solution based on the analogous proof for the affine algebra A_2. Furthermore, we describe all of the modular invariants M that are permutation matrices and prove that there is a 1-1 correspondence between the modular invariants of $C_{r,k}$ and those of $C_{k,r}$. Together with a similar duality of the othogonal algebras $so(n)_k$, this implies that the $C_{2,k}$ classification is actually four classifications in one, and is therefore of particular interest among the affine algebras.

Preface

Conformal field theories are of fundamental interest in mathematics and mathematical physics. In mathematical physics, CFTs are intimately related to string theory, which attempts to provide a unified description of the fundamental forces of our universe. In mathematics, the study of CFTs has influenced abstract algebra, low-dimensional topology, algebraic geometry, and subfactor theory, among others.

The classification of modular invariants is equivalent to the classification of rational conformal field theories (RCFTs), and is therefore of great importance to the physical theory. In this thesis, we are particularly interested in those modular invariants that come from WZW models of RCFTs, as these models have an underlying affine algebra structure. The work in this thesis lays a foundation for the classification of all modular invariant matrices M for the affine Kac-Moody algebra $C_2^{(1)}$ at level k (often abbreviated $C_{2,k}$).

We base this classification on its $A_2^{(1)}$ (equivalently, $sl_3^{(1)}$) counterpart, and specifically on the work done in [10]. In that proof, only the modular invariant axioms ((1.6a) - (1.6c)) were used. Based on these axioms, the simple current and Galois permutations of the integral highest weights of the affine algebra provide symmetries and selection rules for the matrix M.

The next step in the proof of [10] was to classify all of the automorphism invariants, which are defined to be the modular invariant matrices M such that there exists a permutation σ of the set of level k highest weights for which $M_{\lambda\mu} = \delta_{\mu,\sigma(\lambda)}$ for all λ, μ. In the third step, the Galois parity condition was used to greatly reduce the possibilities for M. Finally, all of the exceptional levels were dealt with individually. Our efforts will closely follow this pattern.

All of the preceding terms will be explained in detail in the introductory chapter of this thesis. The second chapter contains the classification of all modular invariants associated to a highest weight $\lambda = (\lambda_1, \lambda_2)$ for which $\lambda_1 = \lambda_2$. Conjecturally, this proof completes about one half of the previously unknown component of the

$C_{2,k}$ classification. The third chapter concerns the duality between the modular invariants M of $C_{r,k}$ and the modular invariants $\tilde{M}$ of $C_{k,r}$. We find that M and $\tilde{M}$ are in a 1-1 correspondence. This rank-level duality is not unique to the C family of affine algebras. For instance, there is also a duality between the orthogonal algebra $so_n^{(1)}$ at level k and $so_k^{(1)}$ at level n. It is well known that so_5 (otherwise known as B_2) is isomorphic to C_2, and this implies that the $C_{2,k}$ classification is actually four classifications in one, namely itself, $C_{k,2}$, $B_{n,5}$ and $D_{n,5}$.

In our fourth and final chapter, we determine all of the automorphism invariants for $C_{2,k}$. The only non-trivial automorphism invariant for $C_{2,k}$ corresponds to the simple current $\mathcal{J}$, and occurs when k is odd. We conjecture that for all odd k, this simple current automorphism invariant is the only nontrivial modular invariant. Our conjecture is given in §1.6.5.

At present, the modular invariant classifications of only two algebras, namely A_1 and A_2, have been determined at arbitrary level. In §1.6.4 we review all of the known affine algebra modular invariant classifications.

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Chapter 1

Introduction

1.1 RCFT and String Theory

The focus of this thesis is the classification of $C_2^{(1)}$ modular invariants. This particular classification is probably of most interest to mathematical physicists, since modular invariants can arise from *conformal field theories* (CFTs) on the torus and they play a role in the classification of vertex operator algebras. A CFT is a conformally invariant, two-dimensional quantum field theory (QFT). As implied by its name, fields are the fundamental objects of a QFT, and can be thought of as operator-valued functions of space-time. A CFT is basically a QFT that is symmetric with respect to conformal (i.e. angle preserving) transformations.

> (T)wo-dimensional conformal field theories are perfect examples of systems in which the symmetries are so powerful as to allow an exact solution to the problem. This feature, as well as the great variety of mathematical concepts needed in their solution and definition, have made conformal field theories one of the most active domains of research in mathematical physics [5].

One physical motivation for studying CFT is *string theory*, where the fundamental object is a string. It is convenient to use strings for descriptive purposes. The two dimensions of a CFT come from the image or path traced by each string, and the (Riemann) surface resulting from the collection of such paths is known as the world-sheet. The physical state of each string is represented by fields that reside on the world-sheet. String dynamics are simple to describe: two strings can fuse into one, or one string can split into two.

In quantum theories, the fundamental numerical quantities are amplitudes, and each Riemann surface is assigned an amplitude (i.e. a complex number). The *vacuum-to-vacuum amplitude* is assigned to a surface without boundary, and in this way the path of each string is quantified. For example, the sphere (genus 0) corresponds to the appearance and dissolution of a single string, while the torus

(genus 1) is used to describe the appearance, splitting, joining, and subsequent dissolution of one string. The initial and end states are both empty, hence the term vacuum.

An algebraic formulation of an RCFT[1] is called a *vertex operator algebra* (VOA). More precisely, any RCFT has two VOAs, each of which has finitely many irreducible modules, which are known as its *primaries*. A VOA is an infinite dimensional graded vector space with bilinear products that obey an infinite number of constraints. It is also a representation of the Virasoro algebra V, which is a one-dimensional extension of the Witt algebra W, satisfying:

$$[L_m\ L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{n,-m} \quad \text{and}$$
$$[L_m\ c] = 0,$$

where $\{L_n \mid n \in \mathbb{Z}\}$ with $c = 0$ forms a basis for W. The grading on the VOA is given by the eigenspaces of the operator L_0, and the central term c acts in V like a scalar multiple of the identity.

It is common to take the two VOAs of an RCFT to be isomorphic. The grading on the vector space induces a grading on the irreducible modules A. Define

$$\mathbb{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\} \tag{1.1}$$

to be the upper half plane. Then if we make the usual substitution $q = e^{2\pi i \tau}$ for $\tau \in \mathbb{H}$, we get the normalised character of A:

$$\chi_A(\tau) = q^{-c/24}\ \operatorname{Tr}_A\ q^{L_0}. \tag{1.2}$$

Strictly speaking, these are the specialised characters $\chi_A(\tau) = \chi_A(\vec{0}, \tau, 0)$, and are linearly dependent. In order to have the definitions in (1.3) make sense, we need to replace χ_A with the corresponding 1-point function. This situation does not arise for the affine algebras, and their full characters are given in (1.36).

If A is an irreducible module for one of the VOAs of an RCFT, then the characters of A have the very special property that under certain transformations of $\mathbb{H}$, χ_A can be written as a sum of characters of the other irreducible modules. This means that we can define matrices S and T by:

$$\chi_A(-\frac{1}{\tau}) = \sum_B S_{AB}\ \chi_B(\tau) \tag{1.3a}$$

$$\chi_A(\tau + 1) = \sum_B T_{AB}\ \chi_B(\tau) \tag{1.3b}$$

where B ranges over all irreducible modules, and $\tau \in \mathbb{H}$. The matrices S and T define a representation of the modular group of the torus (see §1.2), and this representation is known as the modular data of the RCFT (as defined in §1.3).

[1] Rational Conformal Field Theory

We now briefly discuss a specific type of conformal field theory that has an underlying affine algebra structure. Wess-Zumino-Witten (WZW) models of RCFTs correspond to affine Kac-Moody algebras whose level k is a positive integer. The *primary fields* of a WZW model can be identified with the highest weights $\lambda \in P_+^k$ (see §1.6). Among other things, this implies that there exist only a finite number of primary fields. The characters of WZW models can be identified with those of the integrable representations of $X_r^{(1)}$. WZW models are very important examples of conformal field theories, as they are thought to be the building blocks of all RCFTs.

Let H be the space of physical states of a given RCFT. Then we can decompose H into VOA modules [8]:

$$H = \oplus_{A,B}\, M_{AB}\, A \otimes B$$

for irreducible A and B, where the multiplicities are nonnegative integers. The definition of the *torus partition function* in terms of the characters of VOA modules (as in (1.2)) is:

$$\mathcal{Z}(\tau) = q^{-c/24}\bar{q}^{-c/24}\, \mathrm{Tr}_H\, q^{L_0}\bar{q}^{L_0'}. \tag{1.4}$$

Based on the definition of H, (1.4) becomes

$$\mathcal{Z}(\tau) = \sum_{A,B} M_{AB}\, \chi_A(\tau)\, \overline{\chi_B(\tau)}. \tag{1.5}$$

Equation (1.5) is of central importance as it explains how the two VOAs of a conformal field theory should fit together. Whenever we mention the torus (or genus-1) partition function, we will be referring to (1.5).

Recall that string behaviours trace out surfaces called world sheets. If one string splits and subsequently rejoins, then the world sheet is a torus. This '1-loop' contribution to the vacuum-to-vacuum amplitude will look like $\int \mathcal{Z}([\tau])d[\tau]$ where $[\tau]$ is a conformal equivalence class of tori. In the next section, we find a parameterisation of these equivalence classes. In particular, the tori parameterised by τ and $\alpha\tau$ are conformally equivalent for all $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$. Therefore, $\mathcal{Z}(\tau) = \mathcal{Z}(\alpha\tau)$ for all $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$, so that $\mathcal{Z}$ is invariant under the modular group (see section §1.2). Equivalently, the matrix M that defines the partition function is considered to be modular invariant. We discuss modular invariance in greater detail in §1.3.

1.2 The Modular Group

Using standard notation, $\mathrm{SL}_2(\mathbb{Z}) := \{\alpha \in M_2(\mathbb{Z}) \mid \det(\alpha) = 1\}$, and α_{ij} is the element of the matrix α that occupies the ith row and jth column. The nonnegative integers will be represented by $\mathbb{Z}_{\geq}$.

Before discussing modular data in the next section, it is necessary to define the *modular group of the torus*, $\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. The term 'modular group'

has a specific meaning. It indicates that, up to conformal equivalence, every torus is parameterised by an element (or *modulus*) of $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ (for $\mathbb{H}$ as in (1.1)). We will show that this is indeed the case.

We say that two surfaces T_1 and T_2 are *conformally equivalent* if there exists a bijective map $f : T_1 \to T_2$ such that f and f^{-1} are conformal (i.e. preserve angles between curves). A corollary to Abel's Theorem [4] states that any closed genus-1 surface is conformally equivalent to a torus of the form $\mathbb{C}/L$, where L is a 2-dimensional lattice over $\mathbb{Z}$. We can write $L = \mathbb{Z} + \mathbb{Z}\tau$ and restrict τ to $\mathbb{H}$ without consequence (if $\mathrm{Im}(\tau) = 0$, i.e. $\tau \in \mathbb{R}$, then we get a degenerate torus).

The group $\mathrm{PSL}_2(\mathbb{Z})$ acts on the upper-half plane $\mathbb{H}$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \longmapsto \frac{a\tau + b}{c\tau + d}, \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}),\ \tau \in \mathbb{H}$$

Suppose that $\tau \in \mathbb{H}$ and $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$. First note that $\alpha\tau \in \mathbb{H}$, since $\mathrm{Im}(\alpha\tau) = (\mathrm{Im}(\tau))/((\alpha_{21}\mathrm{Re}(\tau) + \alpha_{22})^2 + (\alpha_{21}\mathrm{Im}(\tau))^2) > 0$. The two matrices that generate $\mathrm{SL}_2(\mathbb{Z})$ are $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so it suffices to check that $\mathbb{C}/L$ is conformally invariant under S and T. To make the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ easier to analyse, write $\tau = re^{i\theta}$. Then

$$S : \tau \longmapsto -\frac{1}{r}e^{-i\theta} \quad \text{and}$$
$$T : \tau \longmapsto \tau + 1.$$

The image of the torus $\mathbb{C}/L$ under T is clearly unchanged, and the image of the lattice under S is simply a scaling ($\tau \mapsto 1/\tau$) plus a rotation ($e^{i\theta} \mapsto -e^{-i\theta}$ is a rotation by $(-\theta)$ degrees), both of which are conformal operations. Thus, $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\alpha\tau)$ is conformally equivalent to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ for all $\tau \in \mathbb{H}$ and $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$. This means that in order to exhaust all redundancy, we must restrict τ to $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$; each such τ corresponds to a unique conformal equivalence class of tori. Therefore, $\mathrm{PSL}_2(\mathbb{Z})$ is the modular group of the torus.

Everything that we have just done applies equally well to $\mathrm{SL}_2(\mathbb{Z})$ as it does to $\mathrm{PSL}_2(\mathbb{Z})$. Clearly, $\mathrm{SL}_2(\mathbb{Z})$ can also be described as the modular group of the torus. Throughout this chapter we will refer to *both* $\mathrm{PSL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$ as the modular group, but will distinguish between them when necessary. The reason that we are sometimes more interested in $\mathrm{SL}_2(\mathbb{Z})$ is that the VOA characters in (1.2) define a projective representation of $\mathrm{PSL}_2(\mathbb{Z})$ and a true representation of $\mathrm{SL}_2(\mathbb{Z})$.

1.3 Modular Data

Modular data consists of a finite set Φ of labels, and two matrices S and T indexed by Φ. One of the labels (or primaries) is a special element known as the vacuum,

and is denoted by 0. The matrices must satisfy the following four axioms (where $\overline{S}$ denotes the complex conjugate of S):

- S is unitary and symmetric, T is diagonal and has finite order; (1.6a)
- $S_{0a} > 0 \ \forall a \in \Phi$; (1.6b)
- $S^2 = (ST)^3$; (1.6c)
- $N_{ab}^c := \sum_{d \in \Phi} \frac{S_{ad} S_{bd} \overline{S}_{cd}}{S_{0d}} \in \mathbb{Z}_{\geq}$ (Verlinde's formula). (1.6d)

ıe term **modular data** comes from the fact that S and T give a representation
c ıodular group $\mathrm{SL}_2(\mathbb{Z})$. In order to see that this is the case, define the *fusion*
n ces N_a for each $a \in \Phi$ by:

$$(N_a)_{b,c} = N_{ab}^c = \sum_{d \in \Phi} \frac{S_{ad} S_{bd} \overline{S}_{cd}}{S_{0d}} \tag{1.7}$$

where b and c range over all of Φ. Note that (1.6a) implies that $S\overline{S} = I$ where I is the identity matrix. This gives the useful relationship:

$$\sum_{c \in \Phi} S_{ac} \overline{S}_{cb} = \delta_{a,b}. \tag{1.8}$$

Consider the product of a fusion matrix with the bth column of S. This is a $|\Phi| \times 1$ matrix with dth entry:

$$\begin{aligned}
(N_a S_{\updownarrow,b})_d &= \sum_{c \in \Phi} (N_a)_{d,c} S_{cb} \\
&= \sum_{c \in \Phi} \sum_{e \in \Phi} \frac{S_{ae} S_{de} \overline{S}_{ce}}{S_{0e}} S_{cb} \\
&= \sum_{c \in \Phi} \frac{S_{ab} S_{db} \overline{S}_{cb}}{S_{0b}} S_{bc} + \sum_{e \in \Phi, e \neq b} \frac{S_{ae} S_{de}}{S_{0e}} \sum_{c \in \Phi} \overline{S}_{ec} S_{cb} \\
&= \frac{S_{ab} S_{db}}{S_{0b}} + 0 \quad \text{(by unitarity of } S\text{)} \\
&= \frac{S_{ab}}{S_{0b}} (S_{\updownarrow,b})_d.
\end{aligned}$$

This is equivalent to saying that the bth column of S is an eigenvector for each N_a, with eigenvalue S_{ab}/S_{0b}. We claim that all of these eigenvalues are distinct. If $S_{ab}/S_{0b} = S_{ac}/S_{0c}$ for all $a \in \Phi$ then:

$$\begin{aligned}
&S_{ba} = S_{ab} = \frac{S_{ac} S_{0b}}{S_{0c}} && \text{by hypothesis} \\
&\Rightarrow \delta_{b,c} = \sum_{a \in \Phi} \frac{S_{ac} S_{0b}}{S_{0c}} \overline{S}_{ac} && \text{by (1.8)} \\
&\qquad = \frac{S_{0b}}{S_{0c}} \sum_{a \in \Phi} S_{ca} \overline{S}_{ac} = \frac{S_{0b}}{S_{0c}} > 0 \\
&\Rightarrow \delta_{b,c} = 1
\end{aligned}$$

and so $b = c$. This guarantees that the eigenvalues are linearly independent, and so the columns of S must exhaust all the common eigenvectors of the fusion matrices N_a.

If we take the complex conjugate of $(N_a)_{b,c}$ (whose entries are non-negative integers by (1.6d)) we get:

$$(N_a)_{b,c} = (\overline{N_a})_{b,c} = \sum_{d \in \Phi} \frac{\overline{S}_{ad}\overline{S}_{bd}S_{cd}}{S_{0d}},$$

and so each $\overline{S}_{\updownarrow,b}$ is also an eigenvector for the N_a's. Therefore S and $\overline{S}$ both simultaneously diagonalise the fusion matrices. The simultaneous eigenspaces are all 1-dimensional, so each column of S must also be a column of $\overline{S}$, and vice versa, up to scalar multiplication. This means that there exists a permutation C of Φ, and complex numbers α_b, such that

$$\overline{S}_{ab} = \alpha_b S_{a,Cb}.$$

The matrix S is unitary, so $|\alpha_b| = 1$ for all $b \in \Phi$. If $a = 0$,

$$\overline{S}_{0b} = S_{0b} = \alpha_b S_{0,Cb},$$

and so (1.6b) implies that $\alpha_b > 0$, meaning $\alpha_b = 1$. Therefore, since S is symmetric,

$$\overline{S}_{ab} = S_{a,Cb} = S_{Ca,b}. \tag{1.9}$$

This calculation also tells us that $N_{Ca} = N_a^{\mathrm{T}}$, the transpose of N_a.

We can represent the *conjugation* C as a permutation matrix $\mathcal{C}$ where $(\mathcal{C})_{ab} := \delta_{Ca,b}$. Then (1.9) becomes $\overline{S} = S\mathcal{C}$. We immediately see that $\mathcal{C}$ is an involution:

$$S_{ab} = \overline{\overline{S}}_{ab} = \overline{S}_{a,Cb} = S_{a,C^2b}. \tag{1.10}$$

Combining (1.10) with the definition of $\mathcal{C}$ and the unitary condition $\overline{S} = S^{-1}$, we get $S^{-1} = \overline{S} = S\mathcal{C}$, which implies that $I = S^2\mathcal{C}$. Thus,

$$I = \mathcal{C}^2 = S^4 = (ST)^6. \tag{1.11}$$

The modular data that we are most interested in is that of the affine algebra $C_2^{(1)}$ at level k. In this case, the conjugation C is trivial, and so we have $S^2 = I$.

Finally, we get our representation when we identify

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longmapsto S, \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longmapsto T.$$

Then the matrices on the left generate $\mathrm{SL}_2(\mathbb{Z})$ and obey (1.6c) and (1.11), and we have a representation of the modular group. That is to say, a presentation of $\mathrm{SL}_2(\mathbb{Z})$ is $\langle S,\ T |\ S^2 = (ST)^3,\ S^4 = I \rangle$. In this case our modular group is $\mathrm{SL}_2(\mathbb{Z})$

and not $\mathrm{PSL}_2(\mathbb{Z})$, since in the latter the matrix S must obey $S^2 = I$. Note that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.\tau = -\frac{1}{\tau}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.\tau = \tau + 1$, so our identification makes sense in terms of the character definitions of S and T (as in (1.3) and (1.36)).

Modular data occurs in numerous areas of mathematics [8], but we will restrict our attention to that of affine Lie algebras and RCFT, and in particular to that of $C_2^{(1)}$ at level k (see §1.6 for the definition of affine algebras). Whenever we have modular data, we can consider the set of matrices M that are indexed by Φ and satisfy:

$$\bullet\ MS = SM \text{ and } MT = TM; \tag{1.12a}$$

$$\bullet\ M_{ab} \in \mathbb{Z}_{\geq}\ \forall a, b \in \Phi; \tag{1.12b}$$

$$\bullet\ M_{00} = 1. \tag{1.12c}$$

Any such M is called a *modular invariant*. Modular invariants are central to the CFT classification because they appear in the partition functions $Z(\tau)$ (as defined in (1.5)). In the following sections, we will derive many properties of these matrices that are of interest in the classification of modular invariants of affine algebras, and state several conjectures concerning the $C_{2,k}$ classification. But first, let us make a pair of elementary observations.

The commutativity of the matrices S and M can be written as $SM\overline{S} = M$, since S is unitary and symmetric. In §1.4 we show that $S_{a0} \geq S_{00}$ for all $a \in \Phi$. This gives:

$$1 = M_{00} = \sum_{a,b\in\Phi} S_{0a} M_{ab} S_{b0} \geq S_{00}^2 \sum_{a,b\in\Phi} M_{ab}.$$

Each M_{ab} is a nonnegative integer, and $S_{00}^2 > 0$, so

$$\sum_{a,b\in\Phi} M_{ab} \leq \frac{1}{S_{00}^2}$$

has only finitely many solutions. Hence there are finitely many modular invariants for a given set of modular data.

Another immediate consequence of the modular invariant axioms is the selection rule

$$M_{ab} \neq 0 \Rightarrow T_{aa} = T_{bb}, \tag{1.13}$$

which stems from the fact that T is diagonal.

1.4 Simple Currents

Like the conjugation symmetry in (1.9), *simple current* symmetry provides a useful tool for studying the elements of a modular data S matrix. Recall from §1.3 that the eigenvalues of the fusion matrix N_a (as defined in (1.7)) all have the form

S_{ab}/S_{0b}. The *Perron-Frobenius* eigenvalue of a matrix is the unique, strictly positive eigenvalue τ such that $|\gamma| \leq \tau$ for all other eigenvalues γ [12]. By (1.6b) and the fact that the simultaneous eigenspaces all have dimension equal to one, the Perron-Frobenius eigenvalue of N_a is $\frac{S_{a0}}{S_{00}}$. This means that $|\frac{S_{a0}}{S_{00}}|$ is maximal, and so (using (1.6b))

$$\frac{S_{a0}}{S_{00}} \geq \frac{|S_{ab}|}{S_{0b}}, \text{ i.e. } S_{0a}S_{0b} \geq |S_{ab}|S_{00}. \tag{1.14}$$

Together with the unitarity of S, this implies that $\min_{a\in\Phi} S_{a0} = S_{00}$.

Simple currents are those primaries $a \in \Phi$ for which $S_{a0} = S_{00}$ (the term simple current comes from RCFT). To each simple current $j \in \Phi$, we can associate a phase $\varphi_{\mathcal{J}} : \Phi \to \mathbb{C}$ and a permutation $\mathcal{J}$ of Φ, where $j = \mathcal{J}0$ ('0' is the vacuum). By a minor abuse of notation we also refer to the permutation $\mathcal{J}$ as the simple current. We prove the following simple current symmetry:

$$S_{\mathcal{J}a,b} = \varphi_{\mathcal{J}}(b)S_{ab}. \tag{1.15}$$

Proof. First, note that (1.14) implies that $S_{0b} \geq |S_{jb}|$. By the unitarity of S we have $S_{0b} = |S_{jb}|$. This means that (1.15) holds for $a = 0$, i.e.

$$S_{j,b} = S_{\mathcal{J}0,b} = \varphi_{\mathcal{J}}(b)S_{0b}. \tag{1.16}$$

Consider the product of the fusion matrices N_j and N_{Cj} (where C is the conjugation defined in (1.9)):

$$\begin{aligned}
(N_jN_{Cj})_{ab} &= \sum_{d\in\Phi} (N_j)_{ad}\ (N_{Cj})_{db} \\
&= \sum_{d\in\Phi} \Big(\sum_{e\in\Phi} \frac{S_{je}S_{ae}\overline{S}_{de}}{S_{0e}}\Big)\Big(\sum_{h\in\Phi} \frac{S_{Cj,h}S_{dh}\overline{S}_{bh}}{S_{0h}}\Big) && \text{by (1.6d)} \\
&= \sum_{d\in\Phi} \Big(\sum_{e\in\Phi} \varphi_{\mathcal{J}}(e)S_{ae}\overline{S}_{ed}\Big)\Big(\sum_{h\in\Phi} \overline{\varphi_{\mathcal{J}}(h)}S_{dh}\overline{S}_{hb}\Big) && \text{by (1.16)} \\
&\leq \sum_{d\in\Phi} \delta_{a,d}\ \delta_{d,b} = \delta_{a,b}.
\end{aligned}$$

Therefore both N_j and its inverse $N_{Cj} = N_j^{\mathrm{T}}$ are nonnegative integer matrices. This implies that N_j is orthogonal with exactly one nonzero entry in each row and column, each of which is equal to 1. In other words, N_j is a permutation matrix. This permutation defines $\mathcal{J}$:

$$(N_j)_{ab} = \delta_{b,\mathcal{J}a}.$$

We immediately have $N_{j,a}^{\mathcal{J}a} = (N_j)_{a,\mathcal{J}a} = \delta_{\mathcal{J}a,\mathcal{J}a} = 1$, and so

$$\begin{aligned}
1 = N_{j,a}^{\mathcal{J}a} &= \sum_{d\in\Phi} \frac{S_{jd}S_{ad}\overline{S}_{\mathcal{J}a,d}}{S_{0d}} \\
\Rightarrow \sum_{d\in\Phi} S_{ad}\overline{S}_{da} &= \sum_{d\in\Phi} \varphi_{\mathcal{J}}(d) S_{ad}\overline{S}_{\mathcal{J}a,d} \qquad \text{by (1.8)} \\
\Rightarrow \overline{S}_{ad} &= \varphi_{\mathcal{J}}(d)\overline{S}_{\mathcal{J}a,d} \\
\Rightarrow S_{\mathcal{J}a,d} &= (\overline{\varphi_{\mathcal{J}}(d)})^{-1} S_{ad} = \varphi_{\mathcal{J}}(d) S_{ad} \qquad \text{since } |\varphi_{\mathcal{J}}(b)| = 1.
\end{aligned}$$

□

The simple current $\mathcal{J}$ is a permutation of a finite set, and so it must have finite order. If $\mathcal{J}$ has order n, then by (1.15) it is clear that $\varphi_{\mathcal{J}}$ is an nth root of unity. We can express $\varphi_{\mathcal{J}}$ as:

$$\varphi_{\mathcal{J}}(a) = \exp[2\pi i Q_j(a)] \tag{1.17}$$

where $nQ_{\mathcal{J}}(a)$ is an integer. In later chapters we will use the fact that the affine algebra $C_2^{(1)}$ at level k has a nontrivial simple current with $Q_{\mathcal{J}}(\lambda) = \lambda_1/2$.

We can use simple currents to study the properties of modular invariants. In fact, we can define modular invariants using simple currents. For a simple current $\mathcal{J}$ of order n, define the matrix $M[\mathcal{J}]$ by

$$M[\mathcal{J}]_{ab} = \sum_{\ell=1}^{n} \delta_{\mathcal{J}^\ell a,b}\, \delta(Q_{\mathcal{J}}(a) + \frac{\ell}{2n} r_j)$$

where $\delta(x)$ is 1 if $x \in \mathbb{Z}$ and is 0 otherwise, and r_j is an integer obeying $T_{jj}\overline{T}_{00} = \exp[\pi i r_j(n-1)/n]$. Then M will be a modular invariant if and only if $T_{jj}\overline{T}_{00}$ is an nth root of unity [8]. For $C_{2,k}$ we have $r_j = 4$ and

$$M[\mathcal{J}]_{\lambda\mu} = \begin{cases} \delta_{\mathcal{J}\lambda,\mu} + \delta_{\lambda\mu} & \text{if } \lambda_1 \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{1.18}$$

Let $\mathcal{J}$ and $\mathcal{J}'$ be simple currents. Then, since $M_{\mathcal{J}0,\mathcal{J}'0} \in \mathbb{Z}_{\geq}$, we have:

$$\begin{aligned}
M_{\mathcal{J}0,\mathcal{J}'0} &= \sum_{c,d\in\Phi} S_{\mathcal{J}0,c}\, M_{c,d}\, \overline{S}_{d,\mathcal{J}'0} \\
&= \Big| \sum_{c,d\in\Phi} \varphi_{\mathcal{J}}(c)\, S_{0c}\, M_{cd}\, \overline{\varphi_{\mathcal{J}'}(d)}\, \overline{S}_{d0} \Big| \qquad \text{by (1.15)} \\
&\leq \sum_{c,d} S_{0c} M_{cd} S_{d0} = M_{00} = 1.
\end{aligned}$$

Therefore $M_{\mathcal{J}0,\mathcal{J}'0} \neq 0 \Rightarrow M_{\mathcal{J}0,\mathcal{J}'0} = 1$. This relationship has two important consequences. The first is the selection rule

$$M_{\mathcal{J}0,\mathcal{J}'0} \neq 0 \Rightarrow (M_{ab} \neq 0 \Rightarrow \varphi_{\mathcal{J}}(a) = \varphi_{\mathcal{J}'}(b)). \tag{1.19}$$

This follows from

$$\Big|\sum_{c,d\in\Phi} \varphi_{\mathcal{J}}(c)\,\overline{\varphi_{\mathcal{J}'}(d)}\,S_{0c}M_{cd}\overline{S}_{d0}\Big| \le \sum_{c,d\in\Phi} |\varphi_{\mathcal{J}}(c)\,\overline{\varphi_{\mathcal{J}'}(d)}|\,S_{0c}M_{cd}\overline{S}_{d0} \le \sum_{c,d\in\Phi} S_{0c}M_{cd}\overline{S}_{d0},$$

so that $M_{cd} \neq 0$ implies

$$\sum_{c,d\in\Phi} |\varphi_{\mathcal{J}}(c)\,\overline{\varphi_{\mathcal{J}'}(d)}|\,S_{0c}M_{cd}\overline{S}_{d0} = \sum_{c,d\in\Phi} S_{0c}M_{cd}\overline{S}_{d0}.$$

We must have $|\varphi_{\mathcal{J}}(c)\,\overline{\varphi_{\mathcal{J}'}(d)}| = 1$, and since both $\varphi_{\mathcal{J}}(c)$ and $\overline{\varphi_{\mathcal{J}'}(d)}$ have moduli equal to 1, $\varphi_{\mathcal{J}}(c) = \varphi_{\mathcal{J}'}(d)$. The second consequence is the symmetry

$$M_{\mathcal{J}0,\mathcal{J}'0} \neq 0 \Rightarrow M_{\mathcal{J}a,\mathcal{J}'b} = M_{ab} \quad \forall a,b \in \Phi. \tag{1.20}$$

Equation (1.20) is justified as follows:

$$\begin{aligned} M_{\mathcal{J}a,\mathcal{J}'b} &= \sum_{c,d\in\Phi} S_{\mathcal{J}a,c}\,M_{cd}\,\overline{S}_{d,\mathcal{J}'b} \\ &= \sum_{c,d\in\Phi} \varphi_{\mathcal{J}}(c)\,\overline{\varphi_{\mathcal{J}'}(d)}\,S_{ac}M_{cd}\overline{S}_{db} \\ &= \sum_{c,d\in\Phi} \varphi_{\mathcal{J}}(c)\,\overline{\varphi_{\mathcal{J}'}(c)}\,S_{ac}M_{cd}\overline{S}_{db} \qquad \text{by (1.19)} \\ &= (SM\overline{S})_{ab} = M_{ab}. \end{aligned}$$

So far we have been looking at two simple currents $\mathcal{J}$ and $\mathcal{J}'$, but we are only really interested in one, since $C_{r,k}$ has only one $\mathcal{J}$, and it is an involution. We can show that for any modular invariant associated to $C_r^{(1)}$ at level k, $\varphi_{\mathcal{J}}(a) = \varphi_{\mathcal{J}}(b)$ whenever $M_{ab} \neq 0$ (see §3.3). This implies that the most important selection rule is

$$M_{\mathcal{J}0,\mathcal{J}0} \neq 0 \;\Leftrightarrow\; (M_{ab} \neq 0 \Rightarrow \varphi_{\mathcal{J}}(a) = \varphi_{\mathcal{J}}(b)), \tag{1.21}$$

as it allows us to conclude that $M_{\mathcal{J}0,\mathcal{J}0} = 1$ for all $C_{r,k}$ modular invariants M. We can then combine this with the fact that $M_{\mathcal{J}0,\mathcal{J}0} \neq 0$ implies that $M_{\mathcal{J}'a,\mathcal{J}'b} = M_{ab}$ for all $a, b \in \Phi$ to get $M_{\mathcal{J}0,0} = M_{0,\mathcal{J}0} = M_{\mathcal{J}0,\mathcal{J}0} = M_{00} = 1$.

To see that the only if statement of (1.21) is true, use the proof of (1.20) with $\mathcal{J}' = \mathcal{J}$. Now suppose that $M_{ab} \neq 0 \Rightarrow \varphi_{\mathcal{J}}(a) = \varphi_{\mathcal{J}}(b)$. Then

$$\begin{aligned} M_{\mathcal{J}0,\mathcal{J}0} &= \sum_{c,d\in\Phi} S_{\mathcal{J}0,c}\,M_{cd}\,\overline{S}_{d,\mathcal{J}0} \\ &= \sum_{c,d\in\Phi} \varphi_{\mathcal{J}}(c)\,\overline{\varphi_{\mathcal{J}}(d)}\,S_{0c}M_{cd}\overline{S}_{d0} \\ &= \sum_{c,d\in\Phi} \varphi_{\mathcal{J}}(c)\,\overline{\varphi_{\mathcal{J}}(c)}\,S_{0c}M_{cd}\overline{S}_{d0} \qquad \text{by hypothesis} \\ &= M_{00} = 1. \end{aligned}$$

Therefore we have (1.21). Clearly, the simple current symmetry plays a very important role in the classification of modular invariants.

1.5 Fusion Rings

Like the theory of modular data, the theory of *fusion rings* investigates the consequences of imposing positivity and integrality constraints upon the structure constants N_{ab}^c (as defined in (1.23) below). The fusion ring of an RCFT (possibly associated to an affine algebra) comes from modular data via (1.6d).

More algebraically, a fusion ring is a commutative ring $R = \mathcal{F}(\Phi, N)$ paired with a finite basis Φ over $\mathbb{Q}$, both of which contain the identity element 1, such that:

- The structure constants $N_{ab}^c \geq 0 \ \forall a, b, c \in \Phi$ (1.22a)
- There exists a ring endomorphism $x \mapsto x^*$ stabilising Φ (1.22b)
- $N_{ab}^1 = \delta_{b,a^*}$, (1.22c)

The structure constants determine how the elements of the preferred basis $\{\chi\}$ of R interact:

$$\chi_a \chi_b = \sum_{c \in \Phi} N_{ab}^c \, \chi_c, \tag{1.23}$$

and the ring identity 1 corresponds to the vacuum 0 in modular data.

Whenever we have modular data, we must also have a fusion ring, but the converse does not hold. Modular data can be thought of as a refinement of fusion rings, with the set Φ playing an identical role in both theories. Conspicuously absent, then, are the S and T matrices. While a definition of the latter requires the additional constraint that S and T provide a representation for $\mathrm{SL}_2(\mathbb{Z})$, the former is more straightforward: Choose a basis for the common eigenvectors of the fusion matrices N_a (they are linearly independent and so there will be exactly $|\Phi|$ of them). Normalise this basis and, reordering if necessary, let the first vector be the Perron-Frobenius eigenvector (see §1.4). Then these basis vectors will be the columns of S. This S is unitary, and Verlinde's formula will hold [8].

The ring R is commutative, so (1.22c) implies that the map $x \mapsto x^*$ is an involution. This leads to a two-sided definition of conjugation (the analog of (1.9)). Define matrices $\mathcal{C}_l = SS^t$ and $\mathcal{C}_r = S^tS$, with $(\mathcal{C}_l)_{ab} = \delta_{b,a^*} =: \delta_{b,C_l a}$ defining the permutation C_l corresponding to the matrix $\mathcal{C}_l$. Clearly, both C_l and (the similarly defined) C_r are order two permutations. Hence,

$$\overline{S}_{ab} = S_{C_l a, b} = S_{a, C_r b}.$$

The *dual* of a fusion ring is $\hat{R} = \mathcal{F}(\hat{\Phi}, \hat{N})$, where $\hat{R}$ is the set of all maps $\Phi \to \mathbb{C}$ and has basis $\hat{\Phi}$ consisting of functions $a \mapsto \frac{S_{ai}}{S_{a1}}$. The dual structure constants $\hat{N}_{ab}^c$ are defined as in Verlinde's formula (1.6d) with S replaced by the transpose S^{T}. A fusion ring is *self-dual* if there is a bijection $\iota : \Phi \to \hat{\Phi}$ such that $N_{ab}^c = \hat{N}_{\iota a, \iota b}^{\iota c}$. The

following result solidifies the link between fusion rings and modular data [8].

- The matrix S for which $S_{0a} > 0 \; \forall a \in \Phi$ and the fusion coefficients defined by Verlinde's formula are in $\mathbb{Z}_{\geq}$ is unique up to possible rearrangement of columns.
- R is self-dual $\iff \exists$ bijections $\iota, \iota' : \Phi \to \hat{\Phi}$ such that $S_{a,\iota' b} = S_{b,\iota a}$.

There is an equivalence between a certain class of fusion rings and modular data (without T): a fusion ring with integral fusion coefficients N_{ab}^{c}, self-dual in the strong sense that $\iota = \iota'$, is completely equivalent to a unitary and symmetric matrix S for which $S_{0a} > 0 \; \forall a \in \Phi$ [8].

1.6 Affine Kac-Moody algebras

Affine algebras play a central role in our classification. The standard construction of these infinite dimensional Lie algebras and their resulting representations are described below. As much of what happens is analogous to the finite-dimensional case, we first review the structure and representations of finite-dimensional Lie algebras.

1.6.1 Structure and representations of X_r

Consider a finite-dimensional simple Lie algebra X_r. In other words, X_r is a complex vector space with an anti-symmetric, anti-associative bilinear product $[\ ,\ ]$ and no non-trivial ideals. A *representation* ρ of X_r on a vector space M is a Lie algebra homomorphism from X_r to $gl(M)$, where $gl(M)$ is the Lie algebra of linear maps from M to M with Lie bracket $[f,g] = f.g - g.f$ (for $f, g \in gl(M)$). Equivalently, M is an X_r module with action $x.v = \rho(x).v =: xv$ for any $x \in X_r$ and $v \in M$. Since X_r is a Lie algebra, this action must obey

$$[xy]v = x(yv) - y(xv) \quad \forall x, y \in X_r, \; v \in M$$

and the product xv must be bilinear.

The *adjoint* representation of X_r is especially useful. It is defined as follows:

$$\mathrm{ad} : X_r \to \mathrm{End}(X_r) \tag{1.24}$$

$$\mathrm{ad}(x) : y \mapsto [x, y] \quad \forall y \in X_r. \tag{1.25}$$

We use the adjoint representation to define a symmetric and bilinear form on X_r. The *Killing form* is the trace of the representation:

$$(x|y) = \mathrm{Tr}(\mathrm{ad}(x)\ \mathrm{ad}(y)) \quad \text{for } x, y \in X_r. \tag{1.26}$$

The *Cartan subalgebra* (CSA) of a Lie algebra is a maximal abelian subalgebra $\mathfrak{h}$ such that $\mathrm{ad}x$ is diagonalisable for all x in $\mathfrak{h}$. All such subalgebras are isomorphic; choose one to denote by $\mathfrak{h}$. The CSA of X_r is r-dimensional.

The adjoint representation allows us to decompose X_r as follows [13]:

$$X_r = \mathfrak{h} \oplus \sum_{\alpha} \mathbb{C}x_\alpha$$

where for each $x \in \mathfrak{h}$, $[x, x_\alpha] = \alpha(x)x_\alpha$ for some $\alpha(x) \in \mathbb{C}$. In other words, the $\mathbb{C}x_\alpha$ are simultaneous eigenspaces for $\mathrm{ad}(\mathfrak{h})$. The elements α are maps from $\mathfrak{h}$ to $\mathbb{C}$, so that $\alpha \in \mathfrak{h}^*$. We call these α's the *roots* of the CSA. Let Δ be the complete set of roots for X_r. This root system will always contain a linearly independent subset Π such that any $\alpha \in \Delta$ can be written as a linear combination of the elements of Π with integral coefficients that are all either positive or negative. The elements of Π are called *simple roots* and partition Δ into $\Delta_+ \cup \Delta_-$.

More generally, we can decompose any finite-dimensional X_r module V into $V = \oplus V_\lambda$ where the $\lambda \in \mathfrak{h}^*$ are called *weights* and the

$$V_\lambda = \{v \in V \mid x.v = \lambda(x)v \text{ for } x \in \mathfrak{h}\} \tag{1.27}$$

are called *weight spaces*. Any weight can be written as an integral combination of the *fundamental weights* ω^i for $1 \leq i \leq r$. We identify the weight $\lambda = \lambda_1\omega^1 + \lambda_2\omega^2 + ... + \lambda_r\omega^r$ with the r-tuple $(\lambda_1, \lambda_2, ..., \lambda_r)$. Note that if V is the adjoint module, then λ is not only a weight, but also a root of X_r. For each $x \in \mathfrak{h}$, the *character* of the X_r module V is given by [15]:

$$\mathrm{ch}_V(x) = \sum_{\lambda} (\dim V_\lambda) e^{(\lambda \mid x)}. \tag{1.28}$$

We can create a basis for $\mathfrak{h}$ by defining the *simple coroots* of X_r to be the $\check{\alpha} \in \mathfrak{h}$ such that the Killing form identifies each $\check{\alpha}_i$ with $2\alpha_i/(\alpha_i|\alpha_i) \in \mathfrak{h}^*$ for the simple root α_i. This means that the simple roots act on the simple coroots as:

$$\alpha_j(\check{\alpha}_i) = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} \in \mathbb{Z}, \tag{1.29}$$

where we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ using the Killing form (1.26). The set $\{\check{\alpha}_i\}_{1 \leq i \leq r}$ forms a basis for the CSA. Moreover, the simple coroots are orthogonal to the fundamental weights.

The *Weyl group* W of X_r is the finite group generated by the reflections

$$s_\alpha(\beta) = \beta - 2\frac{(\beta|\alpha)}{(\alpha|\alpha)}\alpha \tag{1.30}$$

where $\beta \in \mathfrak{h}^*$ can be written as a linear combination of the simple roots Π with real coefficients. One of the (many) important properties of W is that each element

$w \in W$ can be written as a product of simple reflections, that is, as a product of some s_{α_i}'s where $\alpha_i \in \Pi$. If the number of these reflections is even, we define $\det(w)$ to be $+1$, otherwise $\det(w) = -1$.

The entire collection of finite-dimensional representations of X_r is well known. Any irreducible finite-dimensional module V of X_r has a highest weight $\lambda = (\lambda_1, ..., \lambda_r)$ with $\lambda_i \in \mathbb{Z}_{\geq}$ for all i. Conversely, to any such λ there is an irreducible finite-dimensional module V. Any finite-dimensional module of X_r can be expressed uniquely as a direct sum of finitely many of these irreducible ones. The Weyl character formula for any simple Lie algebra is [15]:

$$\mathrm{ch}_\lambda(\vec{z}) = e^{-\rho \cdot \vec{z}} \frac{\sum_{w \in W} \det(w) e^{(w(\lambda+\rho) \mid \vec{z})}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha \cdot \vec{z}})} \tag{1.31}$$

where $\vec{z} \in \mathfrak{h}$, Δ_+ is the set of positive roots, $\rho := \frac{1}{2} \sum_{\alpha > 0} \alpha$ is the Weyl vector, and λ is the highest weight of the X_r module V. The so-called *denominator identity* is the equality $\mathrm{ch}_0(\vec{z}) = 1$ put into (1.31) for the trivial 1-dimensional module V_0 (where 0 is the vacuum).

1.6.2 Definition of $X_r^{(1)}$

We are now ready to define $X_r^{(1)}$. By the loop algebra $\mathcal{L}$ of X_r we mean the set of all sums $\sum t^\ell \otimes a_\ell$, $\ell \in \mathbb{Z}$ such that $a_\ell \in X_r$ and all but finitely many $a_\ell = 0$. These sums resemble polynomials in a variable t, and are known as Laurent polynomials. The loop algebra is a Lie algebra with bracket $[t^\ell \otimes a_\ell, t^k \otimes a_k] = t^{\ell+k} \otimes [a_\ell, a_k]$, and its 2-dimensional extension $X_r^{(1)}$ is an affine (non-twisted) Lie algebra:

$$X_r^{(1)} = \mathcal{L} \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where d is the derivation $t\frac{d}{dt}$ and K is the central element. For $x, y \in X_r$ and $a, a_1, b, b_1 \in \mathbb{C}$, the Lie bracket on $X_r^{(1)}$ is given by [14] as:

$$[t^m \otimes x \oplus aK \oplus bd, t^n \otimes y \oplus a_1 K \oplus b_1 d] =$$
$$(t^{m+n} \otimes [x, y] + bnt^n \otimes y - b_1 m t^m \otimes x) \oplus m\delta_{m,-n}(x|y)K.$$

As is typically the case, the central extension is taken to increase the number of available representations, making projective representations of the loop algebra $\mathcal{L}$ into highest weight representations of $X_r^{(1)}$. The derivation is needed to make the weights of $X_r^{(1)}$ appear as distinct linear functionals of the affine CSA, as will be discussed shortly. In this way, the derivation makes the affine Lie algebra behave more like its finite-dimensional counterpart.

For a Kac-Moody algebra, the marks (a_i) and the comarks $(\check{a}_i)$ are the smallest nonnegative integers determined by

$$\sum_{i=0}^{r} a_i \alpha_j(\check{\alpha}_i) = 0 = \sum_{i=0}^{r} \check{a}_i \alpha_i(\check{\alpha}_j) \quad \text{for } j = 0, ..., r. \tag{1.32}$$

The central element K of $X_r^{(1)}$ is defined in terms of the comarks and the simple coroots [15]:

$$K = \sum_{i=0}^{r} a_i \check{\alpha}_i. \tag{1.33}$$

Define

$$\delta := \sum_{i=0}^{r} a_i \alpha_i \qquad \text{and} \qquad \theta := \delta - a_0 \alpha_0 = \sum_{i=1}^{r} a_i \alpha_i. \tag{1.34}$$

We will use (1.33) and (1.34) to connect the structure of $X_r^{(1)}$ with that of X_r. Note that θ is a linear combination of the roots of X_r, while δ is not. Now we can write the affine roots and coroots in terms of Π: If $\Pi = \{\alpha_1, ..., \alpha_r\}$ is the set of simple roots for X_r, then the simple roots for $X_r^{(1)}$ are $\{\alpha_0, \alpha_1, ..., \alpha_r\}$, where $\alpha_0 = a_0^{-1}(\delta - \theta)$ [14]. Similarly, the affine simple coroots are $\{\check{\alpha}_0 = K - a_0\check{\theta}, \check{\alpha}_1, ..., \check{\alpha}_r\}$.

The CSA of the affine algebra $X_r^{(1)}$ has dimension two greater than that of X_r. The extra dimensions are the combined contribution of the central extension and the derivation. From now on we will use $\mathfrak{h}$ to denote the affine CSA.

The affine Weyl group $\hat{W}$ is generated by the *fundamental reflections* s_α of the dual space $\mathfrak{h}^*$. As in the finite-dimensional case, these reflections are defined by (1.30). The essential difference is that the Killing form is positive-definite on $\mathbb{R}\omega^1 + ... + \mathbb{R}\omega^r$ for X_r, but is indefinite on $\mathbb{R}\omega^0 + \mathbb{R}\omega^1 + ... + \mathbb{R}\omega^r$ for $X_r^{(1)}$. W is finite, while $\hat{W}$ is not. We can write $\hat{W}$ as the semi-direct product of W with $\mathbb{Z}^r$ where r is the rank of $X_r^{(1)}$. As suggested by (1.31), Weyl groups appear in the definition of affine modular data.

The algebra $X_r^{(1)}$ is infinite dimensional and its characters for integrable highest weights λ are defined analogously to those of X_r. For the character $\mathrm{ch}_{V(\Lambda)}$ of the integrable highest weight module $V(\Lambda)$ defined as in (1.31), the normalised affine characters relative to the coordinatisation given in [14] look like:

$$\chi_\Lambda(\vec{z}, \tau, u) := e^{2\pi i \tau m_\Lambda} \mathrm{ch}_{V(\Lambda)}, \tag{1.35}$$

where $\tau \in \mathbb{H}$, $\vec{z}$ is in the CSA of X_r, $u \in \mathbb{C}$, and $m_\Lambda := (\Lambda + \rho)^2/2(k + \check{h}) - (\rho)^2/2\check{h}$ ($\check{h}$ is $\sum_{i=0}^{r} \check{a}_i$). The coordinatisation replaces $x \in \mathfrak{h}$ with $(\vec{z}, \tau, u)$ using

$$x = 2\pi i \left(\sum_{i=1}^{\ell} z_i x_i - \tau \Lambda_0 + u\delta \right),$$

where $(\Lambda_0 \,|\, \delta) = 1$, Λ_0 is orthogonal to the root lattice, and the x_i's form an orthonormal basis for $\mathfrak{h}$.

1.6.3 Representations of $X_r^{(1)}$

The analogues of the highest weight representations of X_r are the *integrable highest weight* representations V_Λ. The highest weights for $X_r^{(1)}$ all look like $\Lambda =$

$(\lambda_0, \lambda_1, ..., \lambda_r)$. The term integrable comes from the fact that the highest weight representations of X_r can be integrated up to a projective representation of the loop group $\mathcal{L}$ and hence to a true representation of its central extension $X_r^{(1)}$. We can write any integrable highest weight module V as $V = \bigoplus_{\mu \in \Omega} V_\mu$ where $\Omega = \Omega(\Lambda)$ is the set of weights. For each such V there exists a dominant weight Λ such that $V = V(\Lambda)$. This means that each weight in V can be written as $\Lambda - c$ where either $c = 0$ or c is a nonnegative integral combination of the simple roots of $X_r^{(1)}$ [15]. For a weight $\mu = \Lambda - c$, and element v in V_μ, and $x \in \mathfrak{h}$, the action of x on V_μ is given by (1.27).

A problem arises unless we include a derivation in our definition of $X_r^{(1)}$ [15]. For an integrable highest weight module $V(\Lambda)$ with $\Lambda \neq 0$, each $\Lambda - n\delta$ is a distinct weight for all $n \in \mathbb{Z}_{\geq}$. The simple coroots generate $\mathfrak{h}$, and by (1.32) and (1.34), $\delta(\check{\alpha}_i) = 0$ for all i. This means that $(\Lambda - \delta)(x) = \Lambda(x)\ \forall x \in \mathfrak{h}$. In other words, the weights cannot be distinguished by their action on the CSA. In order to prevent this, we add the derivation d to the CSA and require that $\delta(d) \neq 0$.

The centre of $X_r^{(1)}$ is one-dimensional, and is spanned by K (as given in (1.33)). Define the *level* of a highest weight λ to be the nonnegative integer

$$k := \Lambda(K) = \sum_{i=0}^{r} \check{a}_i \Lambda(\check{\alpha}_i).$$

By (1.32), we have $\delta(K) = 0$. By [14] the normalised characters of Λ and $\Lambda + c\delta$ are equal, and so we will consider $\lambda := \Lambda \bmod \mathbb{C}\delta$ instead of Λ, which amounts to identifying two $X_r^{(1)}$ modules whose characters are the same. Relabelling so that $\lambda_i := \lambda(\check{\alpha}_i)$, we have

$$k = \sum_{i=0}^{r} \lambda_i \check{a}_i,$$

which has only finitely many solutions $(\lambda_0, ..., \lambda_r) \in \mathbb{Z}_{\geq}^{r+1}$ for each positive k, since each $\check{a}_i > 0$. This implies that for each level k, there exist only finitely many λ's such that $V(\lambda)$ has level k. In other words, the integrable highest weight representations of $X_r^{(1)}$ can be partitioned into finite families parametrised by the level k.

Define $P_+^k(X_r^{(1)})$ to be the finite set of level k highest-weights for a given λ, i.e.

$$P_+^k(X_r^{(1)}) = \{\lambda = \lambda_0\omega^1 + ... + \lambda_r\omega^r : \lambda_i \in \mathbb{Z}_{\geq},\ \sum_{i=0}^{r} \lambda_i \check{a}_i = k\}.$$

The *fundamental weights* ω^i for $X_r^{(1)}$ are defined in the same way as those of X_r, except that now their action on the derivation d must be specified. For $i = 0, ..., r$:

$$\begin{aligned} \omega^i(\check{\alpha}_j) &= \delta_{ij} \quad j = 0, ..., r \quad \text{and} \\ (\omega^i, d) &= 0. \end{aligned}$$

In the next subsection we see that P_+^k is the set of primaries for the modular data of $X_r^{(1)}$.

1.6.4 Modular Data for $X_r^{(1)}$

In this section we outline the modular data for affine, non-twisted Kac-Moody algebras $X_r^{(1)}$. As our main interest is in the affine algebra $C_2^{(1)}$ at level k, we state its modular data explicitly in the next subsection.

The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathfrak{h}$ by [14]:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\vec{z}, \tau, u) = \left(\frac{\vec{z}}{c\tau + d}, \frac{a\tau + b}{c\tau + d}, u - \frac{c(\vec{z}|\vec{z})}{2(c\tau + d)} \right).$$

Using the affine characters defined in (1.35), we have, for any level k weight λ:

$$\chi_\lambda\Big(\frac{\vec{z}}{\tau}, -\frac{1}{\tau}, u - \frac{(\vec{z}|\vec{z})}{2\tau}\Big) = \sum_{\mu \in P_+^k} S_{\lambda\mu}\, \chi_\lambda(\vec{z}, \tau, u) \tag{1.36a}$$

$$\chi_\lambda(\vec{z}, \tau + 1, u) = \sum_{\mu \in P_+^k} T_{\lambda\mu}\, \chi_\mu(\vec{z}, \tau, u). \tag{1.36b}$$

Notice that this is similar to the definition of the S and T modular data for RCFT given in (1.3). The difference is that $\vec{z}$ and τ do not have a well-defined meaning for arbitrary RCFTs.

The modular data for any affine Kac-Moody algebra $X_r^{(1)}$ is given in [14] as follows:

$$\Phi = P_+^k(X_r^{(1)});$$

$$0 = k\Lambda_0; \qquad \text{(the vacuum)}$$

$$S_{\lambda\mu} = \frac{i^d}{\kappa^{r/2}\sqrt{|\check{R}|}} \sum_{w \in W} \det(w) \exp\left[-2\pi i \frac{(w(\lambda + \rho)|\mu + \rho)}{\kappa}\right] \tag{1.37}$$

$$T_{\lambda\mu} = \exp\left[-\pi i \frac{(\rho)^2}{\check{h}}\right] \exp\left[\frac{\pi i(\lambda + \rho|\lambda + \rho)}{\kappa}\right] \delta_{\lambda,\mu} \tag{1.38}$$

where $\kappa = k + \check{h}$, $\check{h}$ is the dual Coxeter number $(= \sum_{i=0}^r \check{a}_i)$, $(\ |\ )$ is the Killing form (1.26), $\rho = \sum_{i=0}^r \omega^i$ is the Weyl vector, and W is the finite Weyl group of X_r. In the S matrix definition, d is the number of positive roots and $|\check{R}|$ is the determinant of the coroot lattice. Two of the important formulas which arise from affine algebra modular data are:

$$\frac{S_{\lambda\mu}}{S_{0\mu}} = \mathrm{ch}_{\bar{\lambda}}(-2\pi i \frac{(\bar{\mu} + \rho)}{\kappa}) \tag{1.39}$$

where $\bar{\lambda}$ and $\bar{\mu}$ are weights for X_r and $\mathrm{ch}_{\bar{\lambda}}$ is a finite-dimensional character, and the *Kac-Walton* formula [14], [16]:

$$N_{\lambda\mu}^{\nu} = \sum_{w \in \hat{W}} \det(w) T_{\lambda\mu}^{(w(\nu+\rho)-\rho)}, \tag{1.40}$$

where $\hat{W}$ is the affine Weyl group of $X_r^{(1)}$.

To put our work in context, we list the known affine algebra classifications. They are:

- $A_{1,k}$ by Cappelli-Itzykson-Zuber
- $A_{2,k}$ by Gannon
- $A_{r,1}$ by Degiovanni
- All affine algebras at level 1 by Gannon
- $A_{r,2}$ and $A_{r,3}$ by Gannon
- $B_{r,2}$, $B_{r,3}$, $D_{r,2}$ and $D_{r,3}$ by Gannon
- $A_1 \oplus A_1$ at level (k_1, k_2) by Gannon

1.6.5 Modular Data for $C_2^{(1)}$ at level k

The modular data for $C_2^{(1)}$ at level k is much easier to describe than that of an arbitrary $X_r^{(1)}$. The comarks of $C_{2,k}$ are all equal to one, so the set of level k highest weights is

$$P_+^k = \{\lambda = (\lambda_1, \lambda_2) : \lambda_i \in \mathbb{Z}_{\geq},\ \lambda_1 + \lambda_2 \leq k\}. \tag{1.41}$$

The vacuum is $0 = (k; 0, 0)$, and $\kappa = k + r + 1 = k + 3$. For any λ and μ in P_+^k, the S and T matrices are given by [7]:

$$S_{\lambda\mu} = -\sqrt{\frac{2}{\kappa}} \cdot \det\left[\sin\left(\pi \frac{\lambda[i]\,\mu[j]}{\kappa}\right)\right]_{1 \leq i,j \leq 2} \tag{1.42}$$

and

$$T_{\lambda\lambda} = \exp\left[\frac{-5\pi i}{6}\right] \exp\left[\frac{\pi i(\lambda + \rho | \lambda + \rho)}{\kappa}\right] \tag{1.43}$$

where $\lambda[\ell] := 3 - \ell + \sum_{i=\ell}^{2} \lambda_i$.

In our classification of the $C_{2,k}$ modular invariants we use two important constraints on the modular invariant matrix M, namely the parity condition (see (1.54) below), and the *norm* or T *condition*. Both are powerful tools for reducing the possibilities for M. The norm condition comes directly from the definition of the T matrix given in (1.38). We combine this with the selection rule (1.13) to get

$$M_{\lambda 0} \neq 0 \ \Rightarrow\ (\lambda + \rho)^2 \equiv \rho^2 \pmod{2\kappa}. \tag{1.44}$$

For $C_{2,k}$, we let $a := \lambda_1 + \lambda_2 + 2$ and $b := \lambda_2 + 1$. Then (1.44) becomes

$$a^2 + b^2 \equiv 5 \pmod{4\kappa} \tag{1.45}$$

whenever $M_{\lambda 0} \neq 0$. An immediate consequence of (1.45) is that one of a and b must be odd, while the other is even. Further implications will be discussed in the next chapter. Based on our Theorem in Chapter 2, we state the following conjecture for all $C_{2,k}$ modular invariants M.

Conjecture 1. *For all odd k, $M_{0\lambda} \neq 0 \Rightarrow \lambda = 0$, and for all even k, $M_{0\lambda} \neq 0 \Rightarrow \lambda = 0$ or $\mathcal{J}0$, except for $k = 3, 7, 8, 12$.*

Through computational methods we know this to be true for $k \leq 25,000$. Using (1.5), we list the known $C_{2,k}$ modular invariants:

$\sum \lvert\chi_\lambda\rvert^2$	$\forall k$
$\sum_{\lambda_1 \text{even}} \lvert\chi_\lambda + \chi_{\mathcal{J}\lambda}\rvert^2 + 2\sum_{\lambda_2=0}^{k/2} \lvert\chi_{(k-2\lambda_2,\lambda_2)}\rvert^2$	k even
automorphism invariant	k odd
$\lvert\chi_{00} + \chi_{21}\rvert^2 + \lvert\chi_{03} + \chi_{20}\rvert^2 + 2\lvert\chi_{11}\rvert^2$	$k = 3$
$\lvert\chi_{00} + \chi_{05} + \chi_{22} + \chi_{61}\rvert^2 + \lvert\chi_{02} + \chi_{07} + \chi_{23} + \chi_{60}\rvert^2 + 2\lvert\chi_{31} + \chi_{33}\rvert^2$	$k = 7$
$\lvert\chi_{00} + \chi_{08}\rvert^2 + \lvert\chi_{22} + \chi_{24}\rvert^2 + \lvert\chi_{44} + \chi_{40}\rvert^2 + \lvert\chi_{06} + \chi_{02}\rvert^2 + \lvert\chi_{43} + \chi_{41}\rvert^2 +$ $\lvert\chi_{80}\rvert^2 + \chi_{80}(\chi_{01} + \chi_{07})^* + (\chi_{01} + \chi_{07})\chi_{80}^* + \lvert\chi_{42}\rvert^2 + \chi_{42}(\chi_{25} + \chi_{21})^* +$ $(\chi_{25} + \chi_{21})\chi_{42}^* + \lvert\chi_{04}\rvert^2$	$k = 8$
$\lvert\chi_{00} + \chi_{0,12} + \chi_{23} + \chi_{27} + 2\chi_{44} + \chi_{60} + \chi_{66} + \chi_{81} + \chi_{83}\rvert^2$	$k = 12$

In Chapter 2 we find that under our hypothesis that $\lambda_1 = \lambda_2$, the only non-vacuum weights λ for which $M_{\lambda 0} \neq 0$ occur when $k = 7$ and $k = 12$. These modular invariants correspond to χ_{22}, χ_{44}, and χ_{66}.

1.7 Galois Symmetry

We now investigate the action of certain Galois automorphisms on the elements of the matrix S of any collection of modular data. The resulting Galois symmetry is a generalisation of the conjugation symmetry (1.9) - we can think of the elements of a Galois group as generalisations of complex conjugation.

Let M be the extension of $\mathbb{Q}$ generated by all of the matrix elements S_{ab}, where a and b are in the set of primaries Φ. Then by [3], M is normal with respect to $\mathbb{Q}$, and the Galois group $\mathrm{Gal}(M/\mathbb{Q})$ is abelian. This means, among other things, that M is contained in a cyclotomic field of the form $\mathbb{Q}(\xi_n)$ where $\xi_n = \exp(2\pi i/n)$ is a root of unity. The Galois group $\mathrm{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q})$ can be identified with the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$ of positive integers mod n that are coprime to n. In particular, any element $\alpha \in \mathbb{Q}(\xi_n)$ can be written as a polynomial $p(x)$ with rational coefficients evaluated at $x = \xi_n$. The automorphism σ associated to $\ell \in (\mathbb{Z}/n\mathbb{Z})^*$ obeys $\sigma(\xi_n) = \xi_n^\ell$, hence $\sigma(\alpha) = \sigma(p(\xi_n)) = p(\xi_n^\ell)$.

By way of proving that $\mathrm{Gal}(M/\mathbb{Q})$ is normal, [3] obtains the following Galois symmetry for any $\sigma \in \mathrm{Gal}(M/\mathbb{Q})$:

$$\sigma(S_{ab}) = \epsilon_\sigma(a) S_{\sigma a, b} = \epsilon_\sigma(b) S_{a,\sigma b}, \tag{1.46}$$

where $\epsilon_\sigma : \Phi \mapsto \{\pm 1\}$, and $a \mapsto \sigma a$ is a permutation of Φ. Equation (1.46) has important consequences in the modular invariant classification. In particular, it is used below to derive the parity condition (1.54).

In (1.36) the affine algebra definition of the S matrix is given as:

$$S_{\lambda\mu} = \frac{i^d}{\kappa^{r/2}\sqrt{|\check{R}|}} \sum_{w \in W} \det(w) \exp\left[-2\pi i \frac{(w(\lambda+\rho)|\mu+\rho)}{\kappa}\right]$$

where d is the number of positive roots, r is the rank, and $|\check{R}|$ is the determinant of the coroot lattice (see §1.6). By [3], each element of S can be written as a polynomial over $\mathbb{Q}$ evaluated at a root of unity. In particular, $S_{\lambda\mu} \in \mathbb{Q}(\xi_L)$ for $L = 4\kappa|\check{R}|$. We now have all the tools we need to derive the parity rule for affine algebras.

1.7.1 Parity Condition

Let σ be a Galois automorphism, M a modular invariant and S a modular data matrix associated to the affine algebra $X_r^{(1)}$. Then by (1.12a) of modular invariants, $MS = SM$. By (1.6a) we can write this as $M = SM\overline{S}$, and the automorphism σ can be applied to the matrix M. The entries of M are in $\mathbb{Z} \subset \mathbb{Q}$, so they are fixed by σ. We have:

$$\begin{aligned} M = SM\overline{S} \;&\Rightarrow\; \sigma(M_{\lambda\mu}) = \sigma((SM\overline{S})_{\lambda\mu}) \\ &\Rightarrow M_{\lambda\mu} = \sigma\left(\sum_{\nu,\gamma\in\Phi} S_{\lambda\nu} M_{\nu\gamma} \overline{S}_{\gamma\mu}\right) \\ &= \sum_{\nu,\gamma\in\Phi} \epsilon_\sigma(\lambda)\, S_{\sigma\lambda,\nu}\, M_{\nu\gamma}\, \overline{S}_{\gamma,\sigma\mu}\, \epsilon_\sigma(\mu) \qquad (\mathrm{Gal}(M/\mathbb{Q}) \text{ is abelian}) \\ &= \epsilon_\sigma(\lambda)\epsilon_\sigma(\mu) M_{\sigma\lambda,\sigma\mu}. \end{aligned} \tag{1.47}$$

From (1.47) we get both

$$M_{\lambda\mu} \neq 0 \Rightarrow \epsilon_\sigma(\lambda) = \epsilon_\sigma(\mu) \quad \forall\sigma \tag{1.48}$$

and

$$M_{\sigma\lambda,\sigma\mu} = M_{\lambda\mu}. \tag{1.49}$$

In particular, if we consider only the first column of M, then we must have

$$M_{\lambda 0} \neq 0 \Rightarrow \epsilon_\sigma(\lambda) = \epsilon_\sigma(0). \tag{1.50}$$

This is the basis for our *Galois parity condition.*

By (1.6b) we know that the quotients $S_{\lambda 0}/S_{00}$ are always positive. Applying our automorphism σ yields

$$\sigma\left(\frac{S_{\lambda 0}}{S_{00}}\right) = \frac{\epsilon_\sigma(\lambda) S_{\sigma(\lambda),0}}{\epsilon_\sigma(0) S_{\sigma(0),0}} = \epsilon_\sigma(\lambda)\epsilon_\sigma(0)\left(\frac{S_{\sigma(\lambda),0}}{S_{\sigma(0),0}}\right),$$

and so $\text{sign}\{\sigma(S_{\lambda 0}/S_{00})\} = \epsilon_\sigma(\lambda)\epsilon_\sigma(0)$. The parity condition (1.50) implies that for any nonzero entry in the first row (or column) of M we must have

$$\text{sign}\{\sigma(S_{\lambda 0}/S_{00})\} = 1. \tag{1.51}$$

We can use the Weyl denominator formula (see (1.39) and §1.6.1) to write

$$\frac{S_{\lambda 0}}{S_{00}} = \prod_{\alpha>0} \frac{\sin(\pi(\lambda+\rho)\cdot\alpha/\kappa)}{\sin(\pi\rho\cdot\alpha/\kappa)}. \tag{1.52}$$

Suppose that σ acts on the n^{th} roots of unity by $\sigma(\xi_n) = \xi_n^\ell$ whenever ℓ and n are coprime. Then

$$\sigma(\sin(x)) = \sigma\left(\frac{e^{-ix} - e^{ix}}{2i}\right) = \frac{e^{-i\ell x} - e^{i\ell x}}{2i(-1)^{\frac{\ell-1}{2}}} = (-1)^{\frac{1-\ell}{2}}\sin(\ell x),$$

and so (1.51) becomes

$$\text{sign}\Big\{\prod_{\alpha>0} \frac{\sin(\pi\ell(\lambda+\rho)\cdot\alpha/\kappa)}{\sin(\pi\ell\rho\cdot\alpha/\kappa)}\Big\} = 1. \tag{1.53}$$

Let $\epsilon_\ell(\lambda) := \prod_{\alpha>0}\text{sign}\{\sin(\pi\ell(\lambda+\rho)\cdot\alpha/\kappa)\}$. Then by (1.53), our Galois parity condition is

$$M_{\lambda 0} \neq 0 \Rightarrow \epsilon_\ell(\lambda) = \epsilon_\ell(0). \tag{1.54}$$

In [7], $\epsilon_\ell(\lambda)$ is explicitly calculated for each of the classical algebras. For $C_{2,k}$ and $\lambda = (\lambda_1, \lambda_2) \in P_+^k$, we find that whenever ℓ is coprime to $L = 4\kappa|\check{R}|$,

$$\epsilon_\ell(\lambda) = \text{sign}\left\{\sin\left(\frac{\pi\ell a}{\kappa}\right)\sin\left(\frac{\pi\ell b}{\kappa}\right)\left[\cos\left(\frac{\pi\ell b}{\kappa}\right) - \cos\left(\frac{\pi\ell a}{\kappa}\right)\right]\right\} \tag{1.55}$$

where $a := \lambda_1 + \lambda_2 + 2$ and $b := \lambda_2 + 1$. We will use this form when stating the parity in Chapter 2. For $C_{2,k}$ the determinant of the coroot lattice is a power of two, so we require that ℓ be coprime to 2κ.

We have now developed a sufficient framework for proving our classification theorem.

Chapter 2

Main Results

Theorem 1. *If $M_{\lambda 0} \neq 0$ then $\lambda_1 = \lambda_2$ implies that $\lambda = 0$, except when $k = 7$ or 12. If k equals 7, then $\lambda = (2,2)$, while if $k = 12$ then $\lambda = (4,4)$ or $(6,6)$.*

This theorem tells us that whenever we have a modular invariant matrix M associated to the modular data of the affine algebra $C_{2,k}$ such that $M_{\lambda 0} \neq 0 \Rightarrow \lambda_1 = \lambda_2$, then the only nonzero entry in the first row (or, equivalently, column) of M is $M_{00} = 1$. By [10], Lemma 2, any such modular invariant must be a permutation matrix. In Chapter 3 we show that this further implies that either $M_{\lambda\mu} = \delta_{\lambda,\mu}$ (and so $M = I$) or $M_{\lambda\mu} = \delta_{\mu,\sigma(\lambda)}$ where σ is a permutation of the set Φ and is associated to the simple current $\mathcal{J}$ of $C_{2,k}$.

2.1 Overview

We first review the terminology for $C_{2,k}$. The *level* of our affine algebra is k, but the quantity that most often arises is $\kappa = k + 3$. We find it convenient to write $a := \lambda_1 + \lambda_2 + 2$ and $b := \lambda_2 + 1$ whenever we have a highest weight $\lambda = (\lambda_1, \lambda_2) \in P_+^k$, as defined in (1.41), and we will freely identify (a,b) with λ. Using the (a,b) notation, the vacuum 0 is represented by $(2,1)$, and the constraints imposed by P_+^k translate to $0 < b < a < \kappa$. Recall from (1.55) that the C_2 parity for $a, b \in \mathbb{Z}_{\geq 0}$, with $0 < b < a < \kappa$, is:

$$\epsilon_\ell(a,b) = \operatorname{sign}\left\{\sin\left(\frac{\pi\ell a}{\kappa}\right)\sin\left(\frac{\pi\ell b}{\kappa}\right)\left[\cos\left(\frac{\pi\ell b}{\kappa}\right) - \cos\left(\frac{\pi\ell a}{\kappa}\right)\right]\right\}. \qquad (2.1)$$

The parity condition (1.54) states that $\epsilon_\ell(a,b) = \epsilon_\ell(2,1)$ whenever ℓ is coprime to 2κ. Also recall from (1.45) that the norm condition is

$$a^2 + b^2 \equiv 5 \pmod{4\kappa},$$

which indicates that one of a and b is odd, while the other is even. Under the hypothesis of our theorem, which is equivalent to $a = 2b$, it is clear that b is odd.

The conclusion that $\lambda = 0$ corresponds to $(a, b) = (2, 1)$, and from this point on we will only refer to the independent b parameter, with the assumption that the reader will keep in mind that $a = 2b$. For brevity we will write $\epsilon_\ell(b)$ instead of $\epsilon_\ell(a, b)$.

Note that $\epsilon_\ell(b)$ has an 'ℓ-period' of 2κ, and that $\epsilon_\ell(b) = \epsilon_{\ell b}(1)$. By the latter we know that $\epsilon_\ell(b) = \epsilon_\ell(1)$ if and only if $\epsilon_{\ell b}(1) = \epsilon_\ell(1)$. Hence the Galois parity condition becomes

$$\epsilon_\ell(1) = \epsilon_{\ell b}(1) \quad \forall \ell \text{ coprime to } 2\kappa. \tag{2.2}$$

In the next section we will prove that $\epsilon_\ell(1) = +1$ iff ℓ is in a certain interval $\mathcal{I}$. This result makes (2.2) a powerful tool in narrowing down the possibilities for b.

The idea behind the subsequent proof is to pick an ℓ_0 in $\mathcal{I}$ that is coprime to 2κ, so that the Galois parity condition implies that $\ell_0 b \in \mathcal{I}$. Then by restricting the increment (i.e. the amount by which $\ell_i b$ differs from $\ell_{i+1} b$) to small intervals, we arrive at a contradiction to the parity condition. In this manner we eliminate most of the possibilities. Those few that are not eliminated are investigated numerically using a computer, as detailed in §2.7.

Suppose that we have the prime factorization $\kappa = \prod p^{a_p}$ for distinct primes p with $a_p > 0$. We need the following result:

$$p \mid \kappa,\ p \neq 5 \ \Rightarrow\ b \equiv \pm 1 \pmod{p^{a_p}}. \tag{2.3}$$

Proof. Suppose that $p \mid \kappa$ with multiplicity $a_p > 0$. We write the norm condition in terms of b to get:

$$5b^2 \equiv 5 \pmod{4\kappa}. \tag{2.4}$$

In other words, there exists an integer m such that $5b^2 = 5 + 4m\kappa$. This can be rewritten as $5(b^2 - 1) = 4m\kappa$, so that as long as $p \neq 5$, $p^{a_p} \mid (b+1)(b-1)$. By noting that $(b+1) = (b-1) + 2$, we see that the only prime p that can divide both factors is $p = 2$. Thus it is clear that if p is neither two nor five, $b \equiv \pm 1 \pmod{p^{a_p}}$. The statement is also true when p is two, given the following argument. We know that b is always odd, so both $(b-1)$ and $(b+1)$ are always even, and only one of them is divisible by four. The factor that is divisible by four must also be divisible by 2^{a_2}, since the norm condition actually implies that $4p^{a_p} \mid (b+1)(b-1)$, effectively adding to the multiplicity of p when $p = 2$. □

In our main proof we employ the following strategy. Let q^{a_q} be the maximal prime power divisor of κ, so that $p^{a_p} \leq q^{a_q}$ for all p^{a_p} dividing κ. Then we look at all pairs of κ-divisors (p^{a_p}, r^{a_r}). This is a proof by contradiction, with the hypothesis being that b is not congruent to the same value (± 1) modulo each of p^{a_p}, r^{a_r}, and q^{a_q}. Our conclusion is that this is impossible, so either $b \equiv +1 \pmod{q^{a_q}}$, $b \equiv +1 \pmod{p^{a_p}}$ and $b \equiv +1 \pmod{r^{a_r}}$, or b is congruent to -1 modulo each of these

prime divisors. There is only one q^{a_q} for each κ, and this method eventually implies that either $b \equiv 1 \pmod{p^{a_p}}$ or $b \equiv -1 \pmod{p^{a_p}}$ $\forall p$ dividing κ. Since $0 < b < \kappa/2$, the former possibility means that $b = 1$, while the latter has no solution.

In order to procure the necessary contradictions, we need to find specific ℓ's for which $\epsilon_\ell(1)$ is in $\mathcal{I}$, while $\epsilon_{\ell b}(1)$ is not. The main issue here is that we must have ℓ coprime to 2κ. Given that we need to choose ℓ's of sufficient generality, this is not guaranteed. The question is, when is ℓ_i *not* coprime to 2κ?

In §2.3 we use $\ell_i = 2\kappa i/(p^{a_p}r^{a_r}q^{a_q}) + \ell_0$, where ℓ_0 is coprime to 2κ. Note that ℓ_i is always odd, so $p = 2$ is not a problem. Neither p nor r nor q divides $2\kappa/(p^{a_p}r^{a_r}q^{a_q})$ since p, r and q appear in κ exactly p^{a_p}, r^{a_r} and q^{a_q} times, respectively. Thus it is possible that one of p, r and q divides ℓ_i. Suppose that $p \mid \ell_i$ for some i. Then $\ell_{i+1} = \ell_i + 2\kappa/(p^{a_p}r^{a_r}q^{a_q})$ is not divisible by p. Similarly, each of $\ell_1, ..., \ell_{i-1}, \ell_{i+2}, ..., \ell_{i+(p-1)}$ is not divisible by p. There is also divisibility by r and q to worry about, but if we redefine ℓ_i when one of p, q and r is equal to 3 or 5, then the three smallest primes we have to worry about are 7, 11 and 13. This means that if out of every eight consecutive ℓ_i's there is a minimum of four candidates for which $\epsilon_{\ell_i}(1) \neq \epsilon_{\ell_i b}(1)$, then we are guaranteed that at least one of those ℓ_i's will be coprime to 2κ. The only possible exception to this statement occurs when one of the primes is equal to 7 and both $\ell_1 b$ and $\ell_8 b$ are used. However, this situation does not surface in our proof.

This reasoning is the basis for the proof in §2.3.

2.2 Preliminary results

Before beginning our general proof, we need to know the vacuum parity $\epsilon_\ell(2,1)$ for any ℓ coprime to 2κ, and the number of prime divisors that κ and b can have in common. We find satisfying solutions to both problems.

Define the interval

$$\mathcal{I} \;:=\; \left(\frac{\kappa}{2}, \frac{2\kappa}{3}\right) \cup \left(\frac{4\kappa}{3}, \frac{3\kappa}{2}\right) \;=\; \mathcal{I}_1 \cup \mathcal{I}_2.$$

Then we have:

$$\epsilon_\ell(1) = -1 \quad \text{iff} \quad \ell \in \mathcal{I}. \tag{2.5}$$

To see this result we need only look closely at the Galois parity:

$$\epsilon_\ell(1) = \text{sign}\left\{\sin\left(\frac{2\pi}{\kappa}\right)\sin\left(\frac{\pi}{\kappa}\right)\left[\cos\left(\frac{\pi}{\kappa}\right) + \cos\left(\frac{2\pi}{\kappa}\right)\right]\right\}. \tag{2.6}$$

This parity is very simply determined, as in Table 2.1.

The next result of interest is that whenever b and κ share a prime divisor, that divisor must be equal to 5. This is straightforward, and follows directly from (2.4).

ℓ	$\sin(2\pi\ell/\kappa)$	$\sin(\pi\ell/\kappa)$	$\cos(\pi\ell/\kappa) - \cos(2\pi\ell/\kappa)$	$\epsilon_\ell(1)$
$(0, \frac{\kappa}{3})$	+	+	+	+
$(\frac{\kappa}{2}, \frac{2\kappa}{3})$	−	+	+	−
$(\frac{2\kappa}{3}, \kappa)$	−	+	−	+
$(\kappa, \frac{4\kappa}{3})$	+	−	−	+
$(\frac{4\kappa}{3}, \frac{3\kappa}{2})$	+	−	+	−
$(\frac{3\kappa}{2}, 2\kappa)$	−	−	+	+

Table 2.1: Vacuum parity $\epsilon_\ell(1)$

We will use the following result in the proof of Lemma 1. If ℓ is coprime to 2κ and κ is even, then

$$\ell \in \left(\frac{\kappa}{3}, \frac{2\kappa}{3}\right) \cup \left(\frac{4\kappa}{3}, \frac{5\kappa}{3}\right) \text{ iff } \cos\left(\frac{2\pi\ell b}{\kappa}\right) < \cos\left(\frac{4\pi\ell b}{\kappa}\right). \tag{2.7}$$

To see this, first note that when κ is even, ℓ is coprime to 2κ if and only if $\ell + \kappa$ is coprime to 2κ. This means that we can apply the parity condition to both $\epsilon_\ell(b)$ and $\epsilon_{\ell+\kappa}(b)$. Let J be the union of intervals given in (2.7). Then $I \subset J$ and $\ell \in J$ iff $(\ell + \kappa) \in J$. Furthermore, if $\ell \in J$ then by (2.5), $\epsilon_\ell(1) = -\epsilon_{\ell+\kappa}(1)$. Now the parity condition implies that $\epsilon_\ell(b) = -\epsilon_{\ell+\kappa}(b)$, and we can use the definition for the sign ϵ_ℓ given in (2.1) to show that (2.7) is true.

We now address the possibility that 5 does indeed divide both b and κ.

Lemma 1. *If both b and κ are divisible by 5, then $\kappa = 15$ and $b = 5$.*

Proof. For clarity, the following proof is divided into four sections.
(*i*) First note that $25 \nmid \kappa$. Otherwise, $25 \mid 4\kappa$ and $25 \mid b^2$, so the norm condition (2.4) would imply that $25 \mid 5$.
(*ii*) Suppose that κ is even. Then since our hypothesis guarantees that $\kappa > 10$, we have

$$\frac{\kappa}{2} < \frac{3\kappa}{5} + 1 < \frac{2\kappa}{3},$$

which by (2.5) implies that $\epsilon_\ell(1) = -1$ for $\ell := \frac{3\kappa}{5} + 1$. Recall that b is odd and is divisible by 5. This means that

$$\frac{2\pi\ell b}{\kappa} = \frac{6\pi b}{5} + \frac{2\pi b}{\kappa} \equiv \frac{2\pi b}{\kappa} \pmod{2\kappa}$$

and

$$\frac{\pi\ell b}{\kappa} = \frac{3\pi b}{5} + \frac{\pi b}{\kappa} \equiv \pi + \frac{\pi b}{\kappa} \pmod{2\kappa}.$$

Therefore

$$\epsilon_\ell(b) = \operatorname{sign}\left\{\sin\left(\frac{2\pi b}{\kappa}\right)\sin\left(\frac{\pi b}{\kappa}\right)\left[\cos\left(\frac{\pi b}{\kappa}\right) + \cos\left(\frac{2\pi b}{\kappa}\right)\right]\right\}$$

is the Galois parity for $\ell = 3\kappa/5 + 1$ and the parity condition insists that $\epsilon_\ell(b) = \epsilon_\ell(1) = -1$. However, by (2.7) with $\ell = 1$, we see that $2b + b = 3b < \kappa$, and so

$\cos(2\pi b/\kappa) + \cos(\pi b/\kappa) > 0$. We know that $0 < b < 2b < \kappa$, and so both $\sin(2\pi b/\kappa)$ and $\sin(\pi b/\kappa)$ are positive. Thus $\epsilon_\ell(b) = +1$. This is a contradiction to (2.2) and so we may conclude that κ is odd.

(*iii*) Suppose that $9 \mid \kappa$ and consider $\ell_0 := 2\kappa/3 - 1$. If $9 \mid \kappa$ then this ℓ_0 must be coprime to 2κ. Since $\kappa > 6$, we have $\ell_0 > \kappa/2$, and so $\epsilon_{\ell_0}(1) = -1$. Now let $\ell_i = 2\kappa i/5 + \ell_0$. Under our hypothesis b is divisible by 5 and so

$$\frac{2\pi\ell_i b}{\kappa} = \frac{4\pi i b}{5} + \frac{2\pi\ell_0 b}{\kappa} \equiv \frac{2\pi\ell_0 b}{\kappa} \pmod{2\pi} \quad \text{and}$$
$$\frac{\pi\ell_i b}{\kappa} = \frac{2\pi i b}{5} + \frac{\pi\ell_0 b}{\kappa} \equiv \frac{\pi\ell_0 b}{\kappa} \pmod{2\pi},$$

which together imply that $\epsilon_{\ell_i}(b) = \epsilon_{\ell_0}(b)$.

Now consider ℓ_1 and ℓ_3. By (2.5), both have Galois vacuum parity equal to one:

$$\begin{aligned}\ell_1 &= \frac{2\kappa}{5} + \frac{2\kappa}{3} - 1 > \frac{2\kappa}{3} \\ &= \frac{16\kappa}{15} - 1 < \frac{4\kappa}{3} \qquad \text{and} \\ \ell_3 &= \frac{6\kappa}{5} + \frac{2\kappa}{3} - 1 = \frac{28\kappa}{15} - 1 > \frac{3\kappa}{2}.\end{aligned}$$

At most one of ℓ_1 and ℓ_3 can fail to be coprime to 2κ, since divisibility by 5 is the only issue (recall that ℓ_0 is coprime to 2κ). If $5 \mid \ell_1$, then $\ell_3 = \ell_1 + 4\kappa/5$ is not divisible by 5, and vice-versa. Thus we have $\epsilon_{\ell_i}(b) = \epsilon_{\ell_0}(b) = \epsilon_{\ell_0}(1) = -1$, while $\epsilon_{\ell_i}(1) = +1$ for $i = 1, 3$. This contradicts the parity condition (2.2) and so we must conclude that $9 \nmid \kappa$.

(*iv*) Now suppose that there exists a prime $p > 5$ such that $p \mid \kappa$. Let

$$\ell_0 := \frac{2\kappa}{p}\frac{(p \pm 1)}{4} \pm 1,$$

where we use $(p+1)$ if $p \equiv 3 \pmod 4$ and $(p-1)$ if $p \equiv 1 \pmod 4$, so that $(p \pm 1)/4$ is an integer. Note that because κ is odd, $2\kappa/p$ is not divisible by 4. We use the plus or minus one at the end to ensure that one of our ℓ_0's will be coprime to κ.

Suppose that $p \equiv 3 \pmod 4$. Then

$$\ell_0 := \frac{2\kappa}{p}\frac{(p+1)}{4} \pm 1 = \frac{\kappa}{2} + \frac{\kappa}{2p} \pm 1$$

and so $p > 5$ and $\kappa > 15$ together imply that $\kappa/2p + 1 < \kappa/10 + 1 < \kappa/6$, making $\ell_0 < 2\kappa/3$. Therefore by (2.5), $\epsilon_{\ell_0}(1) = -1$ when $p \equiv 3 \pmod 4$.

Let $\ell_i = 2\kappa i/5 + \ell_0$. Then, as in (*iii*), $\epsilon_{\ell_i}(b) = \epsilon_{\ell_0}(b)$. Also as in the above proof, we consider ℓ_1 and ℓ_3:

$$\ell_1 = \frac{9\kappa}{10} + \frac{\kappa}{2p} \pm 1 < \frac{4\kappa}{3}$$
$$\ell_3 = \frac{17\kappa}{10} + \frac{\kappa}{2p} \pm 1 > \frac{3\kappa}{2}.$$

At least one of these will be coprime to 2κ, and so for $j \in \{1,3\}$,

$$-1 = \epsilon_{\ell_0}(1) = \epsilon_{\ell_i}(1) \neq \epsilon_{\ell_j}(b) = +1,$$

which implies that $p \not\equiv 3 (\mathrm{mod}\ 4)$.

Now consider the possibility that $p \equiv 1 (\mathrm{mod}\ 4)$. Then

$$\ell_0 := \frac{2\kappa}{p}\frac{(p-1)}{4} \pm 1 = \frac{\kappa}{2} - \frac{\kappa}{2p} \pm 1$$

and so $p < \kappa/2$ implies that $3\kappa/2 < \ell_0 < \kappa/2$. Thus by (2.5), $\epsilon_{\ell_0}(1) = +1$ when $p \equiv 1 (\mathrm{mod}\ 4)$. This ℓ_0 needs to be augmented a bit to get a contradiction when $p \equiv 1 (\mathrm{mod}\ 4)$. Let $\ell^* = \ell_0 + \kappa/p + 1$. Then ℓ^* is odd, and will be coprime to κ as long as it is not divisible by p.

Let $\ell_i = 2\kappa i/5 + \ell^*$. Then, as above, $\epsilon_{\ell_i}(b) = \epsilon_{\ell^*}(b)$. Again, we consider ℓ_1 and ℓ_3:

$$\ell_1 = \frac{9\kappa}{10} + \frac{\kappa}{2p} + 1 \pm 1 < \frac{4\kappa}{3}$$
$$\ell_3 = \frac{17\kappa}{10} + \frac{\kappa}{2p} + 1 \pm 1 > \frac{3\kappa}{2}.$$

By (2.5), $\epsilon_{\ell_1}(1)$ and $\epsilon_{\ell_3}(1)$ are both equal to one, and we get the same contradiction that we did when $p \equiv 3 (\mathrm{mod}\ 4)$. Therefore, $p \not\equiv 1 (\mathrm{mod}\ 4)$.

Hence if $p \mid \kappa$ then $p \leq 5$. Our prime p is not equal to two by (*ii*), and each of three and five divide κ exactly once, by (*iii*) and (*iv*), respectively. Thus we are left with $\kappa = 15$ and $b = 5$. □

We conclude this section by combining the two previous results: if $p \mid b$ and $p \mid \kappa$ then $p = 5$, and if $p = 5$ then $\kappa = 15$ and $b = 5$. Therefore,

$$\gcd(b, 2\kappa) = 1 \text{ or } (\kappa, b) = (15, 5). \tag{2.8}$$

2.3 General Proof

For the general proof let us assume that κ has at least three distinct prime divisors (otherwise refer to §2.5 and §2.6). Let q^{a_q} be the maximal prime divisor of κ, i.e. the prime power such that $p^{a_p} < q^{a_q}$ for all $p \mid \kappa$. We will first prove that for all but finitely many κ's, $C \notin (\frac{\kappa}{12}, \frac{23\kappa}{12})$ (the number C is defined below). The next step is to deal with $C \in (0, \frac{\kappa}{12}) \cup (\frac{23\kappa}{12}, 2\kappa)$, as detailed in 2.3.2 and 2.3.3.

Define ℓ_i and the increment X as follows:

$$\ell_i := \frac{2\kappa i}{p^{a_p} r^{a_r} q^{a_q}} + \ell_0 \tag{2.9}$$
$$X := \frac{2\kappa b}{p^{a_p} r^{a_r} q^{a_q}},$$

where

$$\ell_0 = \begin{cases} \frac{\kappa+1}{2} & \text{if } \kappa \equiv 1 (\text{mod } 4) \\ \frac{\kappa+4}{2} & \text{if } \kappa \equiv 2 (\text{mod } 4) \\ \frac{\kappa+3}{2} & \text{if } \kappa \equiv 3 (\text{mod } 4) \\ \frac{\kappa+2}{2} & \text{if } \kappa \equiv 0 (\text{mod } 4) \end{cases} \tag{2.10}$$

is guaranteed to be coprime to 2κ (as long as $3 \nmid \kappa$), and is in $\mathcal{I}_1$ for $\kappa > 12$. If κ is divisible by 3 then we redefine ℓ_i as in §2.4.1. By (2.11) below, κ is always at least 108.

The X defined above is referred to as the increment because it is the amount by which $\ell_i b$ changes as i goes to $i+1$, and let C be the number between 0 and 2κ such that $X \equiv C \pmod{2\kappa}$. Later in this section we get a contradiction for $C \in (\frac{\kappa}{12}, \frac{23\kappa}{12})$ when either $\ell_0 b \in \mathcal{I}_1$ or $\ell_0 b \in \mathcal{I}_2$. This restriction of $\ell_0 b$ is sufficient since ℓ_0 is guaranteed to be coprime to 2κ, and so the parity condition (2.2) will be violated if $\ell_0 b$ is not in $\mathcal{I}$.

Our goal is to find i's for which ℓ_i is in $\mathcal{I}$, while $\ell_i b$ is not, so it is imperative that we be able to determine exactly when the former is true. It turns out that we can restrict ourselves to choosing $\ell_i \in \mathcal{I}_1$, and so we want ℓ_i as defined above to be less than $\frac{2\kappa}{3}$ (ℓ_i is clearly greater than $\frac{\kappa}{2}$). For $\ell_0 = \frac{\kappa}{2} + 2$ (our biggest ℓ_0), this is satisfied when:

$$\begin{aligned} \frac{2\kappa}{3} &> \frac{2\kappa i}{p^{a_p} r^{a_r} q^{a_q}} + \frac{\kappa}{2} + 2 \\ &\Leftrightarrow 2\kappa i < \left(\frac{\kappa}{6} - 2\right) p^{a_p} r^{a_r} q^{a_q} \\ &\Leftrightarrow i < \frac{p^{a_p} r^{a_r} q^{a_q}}{12} - \frac{p^{a_p} r^{a_r} q^{a_q}}{\kappa}. \end{aligned}$$

Here, κ has at least three divisors, and so $(p^{a_p} r^{a_r} q^{a_q})/\kappa \leq 1$. Thus we know that for $i < \frac{1}{12} p^{a_p} r^{a_r} q^{a_q} - 1$, we have ℓ_i in $\mathcal{I}_1$. In the subsequent proof we need ℓ_1 through $\ell_1 1$ in $\mathcal{I}$ and so we require κ to satisfy the following condition:

$$p^{a_p} r^{a_r} q^{a_q} > 144 \text{ for all } \kappa \text{ divisors } p^{a_p}, r^{a_r} < q^{a_q}. \tag{2.11}$$

The remaining possibilities for κ will be covered in §2.7.

2.3.1 $C \in (\frac{\kappa}{12}, \frac{23\kappa}{12})$

As mentioned in 2.1, we are guaranteed that four out of any consecutive eight ℓ_i values cannot all fail to be coprime to three of the divisors of 2κ as long as none of those divisors is 3 or 5. We use this assumption here, and deal with divisibility by 3 and 5 in sections (2.4.1), (2.4.2) and (2.4.3).

Suppose that $C \in (\frac{m\kappa}{12}, \frac{(m+1)\kappa}{12})$ for $m = 1, ..., 22$. Table 2.2 contains information about the locations of the $\ell_i b$'s, where $i = 1, ..., 8$ and ℓ_i is defined as in (2.9). Given the position of $\ell_0 b$ and the restricted range of C, many of the $\ell_i b$'s are forced out of

	$\ell_0 b \in I_1$		$\ell_0 b \in I_2$	
m	(i)	(ii)	(i)	(ii)
1	$-++++$	$++++$	$-++++$	$++++$
2	$++-++$	$+++-+$	$++++$	
3	$++-++$	$++++$	$+++-+$	$++++$
4	$+-+-+$	$++++$	$++-+-+$	$++++$
5	$+-+--+$	$+++-+$	$++-+-+$	$+++-+$
6	$+++--+$		$+-+++$	$+++-+$
7	$++-++$	$++++$	$+-++-+$	$++-+-+$
8	$-++-++$	$++\{-\}-++-+$ or $\{+\}$	$++-++$	$++++$
9	$-+++--+$	$++++$	$+++--+$	
10	$-+-+++$		$++++$	
11	$++-+-+$	$-+-+-+-+$	$+-+\{++\}$	$++++$
	$-+-+-++$	$-+--++$	or $\{-+-+\}$	
12	$+-+\{-+-+\}$	$++++$	$-+++-+$	$++-+-+$
	or $\{++\}$		$-+-+-+-+$	$-+-+-++$
13	$++++$		$-+++$	
14	$+++--+$		$-+++--+$	$++++$
15	$++-++$	$++++$	$-++-++$	$++--++-+$
16	$+-++-+$	$++-+-+$	$++-++$	$++++$
17	$+-+++$	$+++-+$	$+++-+$	$++++$
18	$++-+-+$	$+++-+$	$+-++-+$	$+++-+$
19	$++-+-++$	$++++$	$+-+++$	$++++$
20	$+++$		$++++$	$++-++$
21	$++++$		$++-++$	$+++--+$
22	$-++++$	$++++$	$-+++$	$++++$

Table 2.2: Vacuum parity indicators for the general case $C \in (\frac{m\kappa}{12}, \frac{(m+1)\kappa}{12})$

$\mathcal{I}$. Those that are necessarily *not* in $\mathcal{I}$ are denoted by a '+'. For example, if $\ell_0 b \in \mathcal{I}_1$ and $m = 2$ then $C \in (\frac{\kappa}{6}, \frac{\kappa}{4})$ and so $\ell_1 b$ and $\ell_2 b$ are not in $\mathcal{I}$ and $\ell_3 b \in (\kappa, \frac{17\kappa}{12})$. If $\ell_3 b \notin \mathcal{I}_2$ then neither is ℓ_5 and we are finished (as described in column (ii)), so assume $\ell_3 b \in \mathcal{I}_2 = (\frac{4\kappa}{3}, \frac{3\kappa}{2})$. This implies that $\ell_4 b \in (\frac{3\kappa}{2}, \frac{5\kappa}{3}) \notin \mathcal{I}$, and that $\ell_5 b \notin \mathcal{I}$. The same reasoning is applied to each of the remaining cases.

The chart is divided into two sections, with the first corresponding to the assumption that $\ell_i b \in \mathcal{I}$ the first time such a choice must be made. Once it has been established that there are three $\ell_i b$'s that are not in $\mathcal{I}$, then we assume that each subsequent $\ell_i b$ is in $\mathcal{I}$ until this is not possible. However, this is not necessarily enough to ensure that all the possible outcomes are displayed, and so in some cases parentheses are employed to indicate further possibilities. Specifically, when $m = 8$ and $\ell_0 b \in I_1$, we have similar outcomes regardless of whether $\ell_3 b$ is in $\mathcal{I}$ or not, although these are technically two different cases and are displayed as such.

2.3.2 $C \in (0, \frac{\kappa}{12})$

This particular interval is quite troublesome due to the fact that one of its endpoints is 0. Our solution is to find a better lower bound for C, and then use this information to prove that C cannot be less than $\kappa/12$. The following proof applies to all κ for which (2.11) holds.

Let $b = nq^{a_q} \pm 1$ for some non-negative integer n, where this description is consistent with our original assumption (i.e. if we started by letting $b \equiv 1 \pmod{q^{a_q}}$ then we would use '+1' here). Note that the assumption also implies that b is congruent to $-(\pm 1)$ modulo at least one of r^{a_r} and p^{a_p}, so that n is not divisible by at least one of p^{a_p} and r^{a_r}. Suppose that $n \equiv t \pmod{p^{a_p} r^{a_r}}$. Then, since n is not divisible by $p^{a_p} r^{a_r}$, we have $0 < t < p^{a_p} r^{a_r}$ and

$$C = \frac{2t\kappa}{p^{a_p} r^{a_r}} \pm \frac{2\kappa}{p^{a_p} r^{a_r} q^{a_q}}.$$

To get an improved lower bound for C we note that

$$C < \frac{\kappa}{12 p^{a_p} r^{a_r}} \Rightarrow t < \frac{1}{24} \mp \frac{1}{q^{a_q}} < 1, \tag{2.12}$$

which is a contradiction. Therefore $C > \kappa/(12 p^{a_p} r^{a_r})$. We do a similar calculation to find that $C < \kappa/48$ implies that

$$t \pm \frac{1}{q^{a_q}} < \frac{p^{a_p} r^{a_r}}{96}, \tag{2.13}$$

and so we must have $p^{a_p} r^{a_r} > 82$ since $t \geq 1$ and $q^{a_q} \geq 7$ (otherwise κ is already an exceptional).

For each $L = 2, 3, 4, \ldots$, take $C \in (\frac{\kappa}{2^{L+1} \cdot 3}, \frac{\kappa}{2^L \cdot 3})$. Because $\ell_0 b$ must be in $\mathcal{I}$, we have $\ell_i b = iC + \ell_0 b$ not in $\mathcal{I}$ as long as iC is between $\frac{\kappa}{6}$ and $\frac{2\kappa}{3}$. By (2.12) we need to increase L only until $\kappa/(2^{L+1} \cdot 3) < \kappa/(12 p^{a_p} r^{a_r})$, i.e. until $2^{L-1} > p^{a_p} r^{a_r}$.

Lemma 2. *If $i = 2^L$ then $\ell_i, .., \ell_{i+4} \in \mathcal{I}$, while $\ell_i b, .., \ell_{i+4} b \notin \mathcal{I}$.*

Proof. Let $i = 2^L$. Then $iC \in (\frac{\kappa}{6}, \frac{\kappa}{3})$, and so $\ell_i b$ is not in $\mathcal{I}$ by (2.5). Similarly, $(i+4)C$ is greater than $\kappa/6$ and less than $\kappa/3 + \kappa/(2^{L-2} \cdot 3)$, which is less than or equal to $2\kappa/3$ for $L \geq 2$. Then $(i+1)C$, $(i+2)C$ and $(i+3)$ are stuck in the between the two, and so none of $\ell_i b$, $\ell_{i+1} b$, $\ell_{i+2} b$, $\ell_{i+3} b$ and ℓ_{i+4} is in $\mathcal{I}$.

In order to ensure that ℓ_i, ℓ_{i+1}, ℓ_{i+2}, ℓ_{i+3} and ℓ_{i+4} are in $\mathcal{I}$, we need to know that $2^L + 4 < (p^{a_p} r^{a_r} q^{a_q})/12 - 1$. This is equivalent to

$$p^{a_p} r^{a_r} (\frac{q^{a_q}}{12} - 2) > 5, \tag{2.14}$$

since for each L, $2^{L-1} < p^{a_p} r^{a_r}$ (otherwise we stop). First note that (2.14) is satisfied whenever $q^{a_q} \geq 25$ and $C < \kappa/48$, since by (2.13), we have $p^{a_p} r^{a_r} > 82$, which is sufficient. We can solve $\kappa/48 < C < \kappa/12$ by testing the intervals explicitly. If $q^{a_q} \leq 23$ then κ is an exceptional, and all of the exceptional values are dealt with in §2.7. Therefore we can conclude that

$$2^L + 4 < \frac{p^{a_p} r^{a_r}}{2} + 4 < \frac{p^{a_p} r^{a_r} q^{a_q}}{12} - 1$$

as required. □

Thus we have five consecutive ℓ_i's for which ϵ_{ℓ_i} and $\epsilon_{\ell_i b}$ do not agree. Since one of these is guaranteed to be coprime to 2κ, this is a contradiction and C cannot be less than $\kappa/12$.

2.3.3 $C \in (\frac{23\kappa}{12}, 2\kappa)$

Here again we have a difficult interval, one whose right endpoint is congruent to 0 modulo 2κ. Our strategy will be much the same as in the previous section. In this situation we need a better upper bound for C.

As above, let $n \equiv t \pmod{p^{a_p} r^{a_r}}$. Then

$$C > 2\kappa - \frac{\kappa}{12 p^{a_p} r^{a_r}} \tag{2.15}$$

$$\Rightarrow t > p^{a_p} r^{a_r} \mp \frac{1}{q^{a_q}} - \frac{1}{24}. \tag{2.16}$$

This is a contradiction to the fact that t is an integer that is less than $p^{a_p} r^{a_r}$. Therefore $C < (2\kappa - \kappa/(12 p^{a_p} r^{a_r}))$.

Let $C \in (2\kappa - \frac{\kappa}{2^{L} \cdot 3}, 2\kappa - \frac{\kappa}{2^{L+1} \cdot 3})$ for $L = 2, 3, 4, ...$ Then if $i = 2^L$, $iC \in (2\kappa - \frac{\kappa}{3}, 2\kappa - \frac{\kappa}{6})$. For $\ell_0 b \in \mathcal{I}_1$ this means that $\ell_i b = iC + \ell_0 b \in (\frac{\kappa}{6}, \frac{\kappa}{2})$, while $\ell_0 b \in \mathcal{I}_2$ implies that $\ell_i b \in (\kappa, \frac{4\kappa}{3})$. In either case, $\ell_i b$ is not in $\mathcal{I}$. Similarly, $\ell_{i+1} b$, $\ell_{i+2} b$, $\ell_{i+3} b$ and $\ell_{i+4} b$ are not in $\mathcal{I}$. The only term that is close to coming back into $\mathcal{I}$ is $\ell_{i+4} b$ when $\ell_0 b \in \mathcal{I}_2$:

$$\ell_{i+4} b \in \left(\kappa - \frac{\kappa}{2^{L-2} \cdot 3}, \frac{4\kappa}{3} - \frac{\kappa}{2^{L-1} \cdot 3} \right).$$

Fortunately, $L \geq 2$, and so $\kappa - \frac{\kappa}{2^{L-2}3} \geq \frac{2\kappa}{3}$. Thus none of $\ell_i b, ..., \ell_{i+4} b$ is in $\mathcal{I}$.

The proof that ℓ_i through ℓ_{i+4} are all in $\mathcal{I}_1$ is identical to that of the previous section. Therefore C cannot be an element of the interval $(23\kappa/12, 2\kappa)$.

2.4 Proofs for κ divisible by three or five

We look at the possibility that κ is divisible by 3 or 5 separately. Our strategy will be to apply the general proof found in §2.3 to all of the other prime divisors of κ, and use this information to choose specific ℓ's for which ℓ is in $\mathcal{I}$ while ℓb is not. Note that the proofs found in §2.3.2 and 2.3.3 do not rely on a coprime argument, and so their results will still hold here.

2.4.1 κ divisible by 3

If κ is a multiple of 3, then we must redefine ℓ_i to preclude the possibility that 3 divides ℓ_i. This, together with a similar restriction for ℓ_i when κ is divisible by 5, allows us to conclude that at least one of the ℓ_i's listed in Table 2.2 will be coprime to 2κ.

If, for example, $p = 3$, then our ℓ_i is

$$\ell_i := \frac{2\kappa i}{3^{a_3-1}r^{a_r}q^{a_q}} + \ell_0, \tag{2.17}$$

where $\ell_0 = 4\kappa/3 + 3$ if $a_3 = 1$, and $\ell_0 = 4\kappa/3 + 1$ if $a_3 > 1$. The $a_3 > 0$ version of (2.11) is

$$3^{a_3-1}r^{a_r}q^{a_q} > 144. \tag{2.18}$$

Note that (2.18) will always be satisfied if $a_3 > 3$. When $a_3 \leq 3$ we encounter exceptions to (2.18). If $a_3 = 1$ these exceptions are particularly numerous, and so we opt to deal with the exceptional a_3 values algebraically rather than numerically. The general proof applied to $\kappa/3^{a_3}$ implies that we have two possibilities for b: either $b \equiv 1 \pmod{\kappa/3^{a_3}}$, or $b \equiv -1 \pmod{\kappa/3^{a_3}}$. In the former case (Case I) we assume for contradiction that $b \equiv -1 \pmod{3^{a_3}}$, while for the latter (Case II) we assume that $b \equiv 1 \pmod{3^{a_3}}$. Otherwise $b = 1$ (our goal) and $b = \kappa - 1$ (an impossibility), respectively.

Let us first suppose that $a_3 = 1$, and define $\ell := 2\kappa/3 - 3$. Then this ℓ is in $\mathcal{I}_1$ and is coprime to 2κ ($a_3 = 1$ implies that 3 does not divide $2\kappa/3$). We know that $b < \kappa/2$ and so in Case I, $b = \kappa/3 + 1$. This means that $b + 1$ is divisible by 3, and b odd implies that $b + 1$ is even. Therefore,

$$\begin{aligned} \ell b &= 2\kappa \cdot \frac{(b+1)}{3} - \frac{2\kappa}{3} - 3b \\ &\equiv \frac{4\kappa}{3} - \kappa - 3 \pmod{2\kappa} \\ &= \frac{\kappa}{3} - 3 \notin \mathcal{I}. \end{aligned}$$

a_3	$b \pmod{3^{a_3}}$	ℓ	b	ℓb
2 or 3	-1	$2\kappa/3-1$	$m\kappa/3^{a_3}+1$	$4\kappa/3-b \notin \mathcal{I}$ since $b<\kappa/2$
2	1	$2\kappa/3-1$	$m\kappa/3^{a_3}-1$	$2\kappa/3-b \notin \mathcal{I}$ for $b>\kappa/6$
	1	$5\kappa/9-3$	$\kappa/9-1$	$2\kappa/9-3 \notin \mathcal{I}$
3	1	$5\kappa/9-1$	$m\kappa/27-1$, κ odd	$5\kappa/9-b \notin \mathcal{I}$ for $b>\kappa/18$
	1	$5\kappa/9-2$	$m\kappa/27-1$, κ even	$5\kappa/9-2b \notin \mathcal{I}$ for $b>\kappa/36$
	1	$\kappa/2+8$	$\kappa/27-1$	$43\kappa/54-1 \notin \mathcal{I}$

Table 2.3: ℓ and b values for $1 < a_3 < 4$

In Case II, $b = \kappa/3 - 1$ and $b - 1$ is divisible by 6, so that

$$\begin{aligned} \ell b &= 2\kappa \cdot \frac{(b-1)}{3} + \frac{2\kappa}{3} - 3b \\ &\equiv \frac{5\kappa}{3} + 3 \pmod{2\kappa} \end{aligned}$$

and $\ell b \notin \mathcal{I}$. Thus both cases produce contradictions to the parity condition (2.2), and we may conclude that $a_3 = 1$ implies $b = 1$.

If $a_3 > 1$, let $\ell := 2\kappa/3 - 1$, which is coprime to 2κ and is in $\mathcal{I}_1$. This ℓ works when b is congruent to $-1 \pmod 3$, and if $b \equiv 1 \pmod 3$ and $b > \kappa/6$. There are a few exceptions to the latter situation. The results of all of these cases are summarised in Table 2.3.

For $a_3 = 2$, note that the only b that can be less than $\kappa/6$ is $b = \kappa/9 - 1$. This b is only possible when κ is even, since b must be odd. Thus we can let $\ell := 5\kappa/9 - 3$. If $a_3 = 3$ and κ is odd, then m must be even. This means that the only b value that is not covered in the first two rows of the $a_3 = 3$ section of Table 2.3 is $b = \kappa/27 - 1$. For this b note that:

$$\begin{aligned} b^2 &= \frac{\kappa^2}{3^6} - \frac{2\kappa}{3^3} + 1 \\ &\Rightarrow \frac{\kappa^2}{3^6} - \frac{2\kappa}{3^3} + 1 \equiv 1 \pmod{4\kappa} \quad \text{by (2.4)} \\ &\Rightarrow \kappa\left(\frac{\kappa}{3^6} - \frac{2}{3^3}\right) \equiv 0 \pmod{4\kappa} \end{aligned}$$

and so κ must be congruent to 2 mod 4. This ensure that $\ell := \kappa/2 + 8$ is coprime to 2κ.

Thus $a_3 \leq 3$ implies that there exist ℓ values for which ℓ is in $\mathcal{I}$, while ℓb is not. This contradicts the parity condition (2.2), and so we conclude that $a_3 = 3$ implies that $b = 1$.

2.4.2 κ divisible by 5

In this section, as in the last, we redefine ℓ_i so that it is not divisible by a specific prime. In this case our prime is 5, and the norm condition (2.4) implies that $b \equiv$

$\kappa \pmod 4$	ℓ	b	ℓb
1	$(\kappa+1)/2$	$2\kappa/5+1$	$17\kappa/10+1/2$
	$(\kappa+1)/2$	$2\kappa/5-1$	$7\kappa/10-1/2$
2	$(\kappa+4)/2$	$\kappa/5+1$	$19\kappa/10+2$
	$(\kappa+4)/2$	$\kappa/5-1$	$9\kappa/10-2$
	$(\kappa+4)/2$	$2\kappa/5+1$	$13\kappa/10+2$
	$(\kappa+4)/2$	$2\kappa/5-1$	$3\kappa/10-2$
3	$(\kappa+3)/2$	$2\kappa/5+1$	$\kappa/10+3/2$
	$(\kappa+3)/2$	$2\kappa/5-1$	$11\kappa/10-3/2$
0	$(\kappa+2)/2$	$\kappa/5+1$	$7\kappa/10+1$
	$(\kappa+2)/2$	$\kappa/5-1$	$17\kappa/10-1$
	$(\kappa+2)/2$	$2\kappa/5+1$	$9\kappa/10+1$
	$(\kappa+2)/2$	$2\kappa/5-1$	$19\kappa/10-1$

Table 2.4: ℓ and b values for $a_5 = 1$

$\pm 1 \pmod{5^{a_5-1}}$. The general proof (§2.3) applies for all the other prime divisors of κ, but we need a special ℓ_i when one of p, r or q is 5. This is achieved by decreasing a_5 by one. For example, if $p = 5$ then

$$\ell_i := \frac{2\kappa i}{5^{a_5-1}r^{a_r}q^{a_q}} + \ell_0, \tag{2.19}$$

where ℓ_0 is unchanged from (2.10). Then in order to have $\ell_1, ..., \ell_8$ in $\mathcal{I}$, we need $5^{a_5-1}r^{a_r}q^{a_q} > 144$. This is guaranteed if $a_5 \geq 3$, but if $a_5 \leq 2$ it is possible that there are κ's for which this condition will not hold. In fact, there are κ's of fairly large magnitude which become exceptionals in this case. To remedy the situation we compute the possible b values for $a_5 \leq 2$ directly.

Let us suppose that κ is divisible by 5 and that $a_5 = 1$. Then (2.4) implies that $b^2 \equiv 1 \pmod{\kappa/5}$, so that we can write $b = m\kappa/5 \pm 1$. By our hypothesis, $b < \kappa/2$, and so m must be one of $\{0, 1, 2\}$. If $m = 0$ then $b = \pm 1$. If κ is odd then $m = 1$ implies b is even and so the only case to consider is $m = 2$, i.e.

$$b = \frac{2\kappa}{5} \pm 1.$$

If κ is even, then both $m = 1$ and $m = 2$ are possibilities.

For each value of $\kappa \pmod 4$, an appropriate ℓ is chosen that will always be coprime to 2κ. In every case, (see Table 2.4) ℓ is in $\mathcal{I}$ but ℓb is not. Therefore $a_5 = 1$ implies that $b = 1$.

If $a_5 > 1$ then when we apply the general proof to the prime power divisors of $\kappa/5^{a_5}$ our conclusion is that $b \equiv \pm 1 \pmod{\kappa/5^{a_5}}$. If $b \equiv 1 \pmod{\kappa/5^{a_5}}$ then we assume for contradiction that $b \equiv -1 \pmod{5^{a_5-1}}$. Since $a_5 = 2$, we have

$$b = \frac{m\kappa}{25} + 1, \qquad m \in \mathbb{Z}. \tag{2.20}$$

As always, $b < \kappa/2$, and so $m \in \{0, ..., 12\}$. If κ is even, let $\ell := 3\kappa/5 - 1$ which is coprime to 2κ and in $\mathcal{I}_1$. Then

$$\begin{aligned} \ell b &= 3\kappa \cdot \frac{(b+1)}{5} - \frac{3\kappa}{5} - b \\ &\equiv \frac{7\kappa}{5} - b \pmod{2\kappa} \\ &= \frac{4\kappa}{3} + \frac{\kappa}{15} - b \end{aligned}$$

and so we need $b > \kappa/15$ if ℓb is to be out of $\mathcal{I}$. By (2.20), ℓb is necessarily not in $\mathcal{I}$ unless $b = \kappa/25 - 1$.

If κ is odd, we use $\ell := 3\kappa/5 + 2$ which is also in $\mathcal{I}_1$ and is coprime to 2κ. Then

$$\begin{aligned} \ell b &= 3\kappa \cdot \frac{(b+1)}{5} - \frac{3\kappa}{5} + 2b \\ &\equiv \frac{4\kappa}{3} + \frac{\kappa}{15} + 2b \pmod{2\kappa} \end{aligned}$$

which is not in $\mathcal{I}$ when $b > \kappa/20$. As above, the only exception to this is $b = \kappa/25 - 1$.

We now assume that $b \equiv -1 \pmod{\kappa/5^{a_5}}$ and that $b \equiv 1 \pmod{5^{a_5 - 1}}$. If κ is even let $\ell := 3\kappa/5 + 1$, so that ℓ is coprime to 2κ and in $\mathcal{I}_1$. Then

$$\begin{aligned} \ell b &= 3\kappa \cdot \frac{(b-1)}{5} + \frac{3\kappa}{5} + b \\ &\equiv \frac{2\kappa}{3} - \frac{\kappa}{15} + b \pmod{2\kappa} \end{aligned}$$

so that $b > \kappa/15$ will guarantee that $\ell b \notin \mathcal{I}$.

If κ is odd, let $\ell := 3\kappa/5 - 2 \in \mathcal{I}_1$. Then

$$\begin{aligned} \ell b &= 3\kappa \cdot \frac{(b-1)}{5} + \frac{3\kappa}{5} - 2b \\ &\equiv \frac{2\kappa}{3} - \frac{\kappa}{15} - 2b \pmod{2\kappa} \end{aligned}$$

for which b must be bigger than $\kappa/20$ to force ℓb out of $\mathcal{I}$.

In each of the four cases described above, the only b values for which we do not get a contradiction are $b = \kappa/25 \pm 1$. Note that for these b's we have:

$$\begin{aligned} b^2 &= \frac{\kappa^2}{5^4} \pm \frac{2\kappa}{5^2} + 1 \\ &\Rightarrow 5b^2 = \frac{\kappa^2}{5^3} \pm \frac{2\kappa}{5} + 5 \\ &\Rightarrow \frac{\kappa^2}{5^3} \pm \frac{2\kappa}{5} + 5 \equiv 5 \pmod{4\kappa} \quad \text{by (2.4)} \\ &\Rightarrow \kappa\left(\frac{\kappa}{5^3} \pm \frac{2}{5}\right) \equiv 0 \pmod{4\kappa} \end{aligned}$$

and so κ must be congruent to 2 mod 4. If $b = \kappa/25 + 1$, this implies that $b \equiv 3 \pmod 4$, while if $b = \kappa/25 - 1$, we have $b \equiv 1 \pmod 4$. For the former b, let $\ell := \kappa/2 + 2 \in \mathcal{I}_1$. Then

$$\begin{aligned} \ell b &= \kappa \cdot \frac{(b+1)}{2} - \frac{\kappa}{2} + 2b \\ &\equiv \frac{3\kappa}{2} + \frac{2\kappa}{25} + 2 \pmod{2\kappa} \end{aligned}$$

and so ℓb is not in $\mathcal{I}$. For the latter case, $\ell := \kappa/2 + 8 \in \mathcal{I}_1$:

$$\begin{aligned} \ell b &= \kappa \cdot \frac{(b-1)}{2} + \frac{\kappa}{2} + 8b \\ &\equiv \frac{\kappa}{2} + \frac{8\kappa}{25} - 8 \pmod{2\kappa} \\ &= \frac{41\kappa}{50} - 8 \end{aligned}$$

and, since ℓb is clearly bigger than $2\kappa/3$, we are finished.

When $a_5 > 2$, we use the general proof in §2.3 to find that either $b \equiv -1 \pmod{5^{a_5-1}}$ and $b \equiv -1 \pmod{p^{a_p}}$ for all $p \mid \kappa$, $p \neq 5$, or $b \equiv 1 \pmod{5^{a_5-1}}$ and $b \equiv 1 \pmod{p^{a_p}}$ for all p. In the former case, note that $\kappa/5^{a_5}$ and 5^{a_5-1} are coprime, and so $b+1$ is divisible by $(\kappa/5^{a_5}) \cdot 5^{a_5-1} = \kappa/5$. Similarly, the latter implies that $b \equiv 1 \pmod{\kappa/5}$. Therefore, the $a_5 > 2$ case reduces to the $a_5 = 1$ case, and, as we have dealt with the $a_5 = 2$ case explicitly, $C \notin (\kappa/12, 23\kappa/12)$.

2.4.3 κ divisible by 15

We now modify the $a_3 > 0$ proof to include the possibility that 5 divides κ. As before, the general proof applied to all other prime power divisors of κ yields $b \equiv \pm 1 \pmod{\kappa/3^{a_3}5^{a_5}}$. We know from (2.3) and (2.4) that $b \equiv \pm 1 \pmod{3^{a_3}}$ and $b \equiv \pm 1 \pmod{5^{a_5-1}}$. If $b \pmod{\kappa/3^{a_3}5^{a_5-1}}$ and $b \pmod{3^{a_3}}$ are the same, then

$$b \equiv \pm 1 \pmod{\kappa/5^{a_5}},$$

and we can use the standard $a_5 > 0$ proof with a new ℓ_0: $\ell_0 := 4\kappa/3 + 3$ if $a_3 = 1$, while $\ell_0 := 4\kappa/3 + 1$ if $a_3 > 1$. If b is congruent to the same value $\pmod{\kappa/3^{a_3}5^{a_5}}$ as it is $\pmod{5^{a_5-1}}$ then

$$b \equiv \pm 1 \pmod{\kappa/3^{a_3}5}$$

and we have to augment the existing $a_3 > 0$ proof (we call this Case A). The final situation to consider is that $b \equiv \pm 1 \pmod{3^{a_3}5^{a_5-1}}$ (Case B). Note that the two cases are identical when $a_5 = 1$.

Let us consider Case A first. Note that the value of a_5 does not affect any of the conditions on b. As detailed in the $a_3 > 0$ section, we may restrict ourselves to $1 \leq a_3 \leq 3$. Table 2.5 details the different possibilities for b and the ℓ values which are chosen to contradict the parity condition.

a_3	$b \pmod{3^{a_3}}$	ℓ	b	$\ell b \pmod{2\kappa}$
1	∓ 1	$2\kappa/3-3$	$m\kappa/15 \pm 1$	$4\kappa/3 - 3b \notin \mathcal{I}$ unless $m=4$
		$2\kappa/3-9$	$4\kappa/15 \pm 1$	$14\kappa/15 \mp 9 \notin \mathcal{I}$
2 or 3	-1	$2\kappa/3-1$	$m\kappa/3^{a_3}5+1$	$4\kappa/3 - b \notin \mathcal{I}$
2 or 3	1	$2\kappa/3-1$	$m\kappa/3^{a_3}5-1$	$2\kappa/3 - b \notin \mathcal{I}$ when $b > \kappa/6$
2	(κ even)	$5\kappa/9-3$	$m\kappa/45-1$	$5\kappa/9 - 3b \notin \mathcal{I}$ when $b < \kappa/6$
	(κ odd)	$5\kappa/9-6$	$m\kappa/45-1$	$5\kappa/9 - 3b \notin \mathcal{I}$ when $b < \kappa/6$
3	1	$14\kappa/27-3$	$m\kappa/135-1$	$14\kappa/27 - 3b \notin \mathcal{I}$ when $b < \kappa/6$

Table 2.5: ℓ and b values for $a_3a_5 > 0$: Case A

a_3	$b \pmod{3^{a_3}}$	ℓ	b	$\ell b \pmod{2\kappa}$
1	∓ 1	$2\kappa/3-3$	$m\kappa/75 \pm 1$	$4\kappa/3 - 3b \notin \mathcal{I}$ unless $17 \le m \le 20$
		$2\kappa/3-9$	$m\kappa/75 \pm 1$	$4\kappa/3 - 9b \notin \mathcal{I}$ when $17 \le m \le 20$
2 or 3	-1	$2\kappa/3-1$	$m\kappa/3^{a_3}5^2+1$	$4\kappa/3 - b \notin \mathcal{I}$
2 or 3	1	$2\kappa/3-1$	$m\kappa/3^{a_3}5^2-1$	$2\kappa/3 - b \notin \mathcal{I}$ when $b > \kappa/6$
2	(κ even)	$5\kappa/9-3$	$m\kappa/225-1$	$5\kappa/9 - 3b \notin \mathcal{I}$ when $m > 4$
		$5\kappa/9-9$	$m\kappa/225-1$	$5\kappa/9 - 9b \notin \mathcal{I}$ when $m > 1$
		$5\kappa/9-27$	$\kappa/225-1$	$289\kappa/666 + 27 \notin \mathcal{I}$
	(κ odd)	$5\kappa/9-6$	$m\kappa/225-1$	$5\kappa/9 - 6b \notin \mathcal{I}$ when $m > 2$
		$5\kappa/9-18$	$m\kappa/225-1$	$5\kappa/9 - 12b \notin \mathcal{I}$ when $m \le 2$
3	1	$14\kappa/27-3$	$m\kappa/675-1$	$14\kappa/27 - 3b \notin \mathcal{I}$ when $m > 3$
		$14\kappa/27-9$	$m\kappa/675-1$	$14\kappa/27 - 9b \notin \mathcal{I}$ when $m > 1$
		$14\kappa/27-27$	$m\kappa/675-1$	$323\kappa/675 + 27 \notin \mathcal{I}$

Table 2.6: ℓ and b values for $a_3a_5 > 0$: Case B

In each case ℓ is coprime to 2κ for the given value of a_3. We also want ℓ to be in $\mathcal{I}$, so we require κ to be at least 162 (so that $\ell = 14\kappa/27 - 3 > \kappa/2$).

In Case B our method is similar: find a specific ℓ for each value of a_3 so that $\ell \in \mathcal{I}$ while ℓb is not. Now we have $b \equiv \pm 1 \pmod{3^{a_3}5^{a_5-1}}$ while $b \equiv \mp 1 \pmod{\kappa/3^{a_3}5^{a_5}}$. When a_5 is greater than 2, κ no longer needs to be dealt with in this manner (we can apply the general proof to the ℓ_i given in (2.19) to find that $b \equiv \pm 1 \pmod{\kappa/3^{a_3}5}$ and so Case A applies). Thus we restrict to $a_3 \le 3$ and $a_5 = 2$. The results are shown in Table 2.6. Note that we require $\kappa > 1458$ in order to have $\ell = 14\kappa/27 - 27$ in $\mathcal{I}_1$.

2.5 $\kappa = p^{a_p}$

The proofs in this section are very straightforward, especially when κ is not a power of 5. As the previous proofs do not apply when κ has only one prime divisor, we deal with this possibility directly.

First suppose that $p \neq 5$. Then we know that $b \equiv \pm 1 \pmod{p^{a_p}}$. This is exactly the situation described near the end of §2.1. Therefore, $\kappa = p^{a_p}$ and $p \neq 5$ together imply that $b = 1$.

Now we deal with the case where $p = 5$. If $a_5 = 1$, then $\kappa = 5$, and so $b = 1$ since b is odd and less than $\kappa/2$. Now assume that $a_5 > 1$, so that $5b^2 = 5 + 4t \cdot 5^{a_5}$ for some $t \in \mathbb{Z}_{\geq 0}$. This implies that $b^2 = 1 + 4t \cdot 5^{a_5 - 1}$, i.e.

$$b^2 \equiv 1 \pmod{5^{a_5-1}}. \tag{2.21}$$

We have two possibilities, the first of which is that $b \equiv 1 \pmod{5^{a_5-1}}$. This means that $b = m\frac{5}{5} + 1$ for some $m \in \mathbb{Z}_{\geq 0}$. Now,

$$\begin{aligned} b < \frac{\kappa}{2} &\Rightarrow m\frac{\kappa}{5} + 1 < \frac{\kappa}{2} \\ &\Rightarrow m < \frac{5}{2} - \frac{5}{\kappa} < \frac{5}{2}, \end{aligned}$$

and so m must be one of $\{0, 1, 2\}$. If $m = 1$ then $b = 5^{a_5-1} + 1$ is even, contradicting (2.4).

Thus either $m = 2$, and so $b = 2 \cdot 5^{a_5-1} + 1$, or $b = 1$ (assume the former). Now define $\ell := (\kappa + 1)/2$. This ℓ is in $\mathcal{I}_1$ and is coprime to 2κ. Since 5 is the only prime divisor of κ and $5 \nmid (\kappa + 1)$, 5 does not divide $(\kappa + 1)/2 = \ell$. We have

$$\begin{aligned} \ell b &= \left(\frac{\kappa}{2} + \frac{1}{2}\right)\left(\frac{2\kappa}{5} + 1\right) = \frac{\kappa^2}{5} + \frac{7\kappa}{10} + \frac{1}{2}, \text{ and} \\ \frac{\kappa^2}{5} &= (\frac{\kappa}{5} - 1)\kappa + \kappa \equiv \kappa \pmod{2\kappa}, \text{ so that} \\ \ell b &\equiv \frac{17\kappa}{10} + \frac{1}{2} \pmod{2\kappa}. \end{aligned}$$

This ℓb is between $3\kappa/2$ and 2κ, and so $\ell b \notin \mathcal{I}$ by (2.5). This is a contradiction, and therefore $b = 1$.

The second possibility is that $b \equiv -1 \pmod{5^{a_5-1}}$. Then $b = m\kappa/5 - 1$ and $m \in \{1, 2\}$. As in the proof above we must have $m = 2$, i.e. $b = 2\kappa/5 - 1$. For ℓ as before, we get:

$$\ell b = \left(\frac{\kappa}{2} + \frac{1}{2}\right)\left(\frac{2\kappa}{5} - 1\right) \equiv \frac{7\kappa}{10} - \frac{1}{2} \pmod{2\kappa}.$$

Note that ℓb will not be in the interval $\mathcal{I}$ as long as $7\kappa/10 - 1/2$ is greater than $2\kappa/3$. This condition is equivalent to $\kappa > 15$, and is satisfied when $a_5 > 1$. Thus b cannot be congruent to negative 1 modulo 5^{a_5-1}. Our conclusion is that $\kappa = p^{a_p}$ implies that $b = 1$.

2.6 $\kappa = p^{a_p}q^{a_q}$

When κ is the product of two prime powers, we define ℓ_i as follows:

$$\begin{aligned} \ell_i &:= \frac{2\kappa i}{p^{a_p}q^{a_q}} + \ell_0, \quad \text{and} \\ X &:= \frac{2\kappa b}{p^{a_p}q^{a_q}}, \end{aligned}$$

where ℓ_0 is unchanged from (2.10). Then ℓ_i is guaranteed to be in $\mathcal{I}$ as long as $i < (p^{a_p}q^{a_q})/12-1$. We will reference the general proof, so we need $\kappa = p^{a_p}q^{a_q} > 144$. If κ is less than or equal to 144, then it is an exceptional, and is dealt with in the final section of this chapter.

Close examination of the general proofs reveals that all of the results are translatable to the two prime case. The exceptionals from §2.3.2 and §2.3.3 will change, but we may appropriate the entire set of results from §2.3. To see this, simply let $r^{a_r} = 1$.

If one of p and q is equal to 3 or 5, we apply the appropriate proofs from §2.4. The redefined ℓ_i's are produced by setting $r^{a_r} = 1$ in (2.17) and (2.19), respectively. Our conclusion from §2.3 remains that $b \equiv \pm 1 \pmod{\kappa/5}$, and so we must invoke §2.4.2.

The two-prime analogues of §2.3.2 and §2.3.3 are simpler than their predecessors. If $b \equiv \pm 1 \pmod{q^{a_q}}$ and $n \equiv t \pmod{p^{a_p}}$ then $0 < t < 1$ and

$$\begin{aligned} C < \frac{\kappa}{12} &\Rightarrow \frac{2t\kappa}{p^{a_p}} \pm \frac{2\kappa}{p^{a_p}q^{a_q}} < \frac{\kappa}{12} \\ &\Rightarrow \frac{t}{p^{a_p}} \pm \frac{1}{p^{a_p}q^{a_q}} < \frac{1}{24} \\ &\Rightarrow t < \frac{p^{a_p}}{24} \mp \frac{1}{q^{a_q}}. \end{aligned}$$

Thus $t \geq 1$ implies that $p^{a_p}/24 \mp 1/q^{a_q} > 1$, and so either $p^{a_p} > 24$ or $p^{a_p} > 22$. In either case, $q^{a_q} \geq 25$ and so p^{a_p} and q^{a_q} are big enough to ensure that the general proof found in §2.3.2, in particular (2.14), will apply without further exceptionals. A similar argument produces the same result when we assume that $C > 23\kappa/12$.

2.7 Exceptional Levels

Here, as in the other sections of this chapter, we refer to those κ's to which our proof does not apply as the exceptional levels of $C_{2,k}$.

Under the hypothesis of our theorem, the exceptional κ's come from three sources, namely (2.5), §2.4.3, and (2.14). The first two sets arise in the proof of the general case of the theorem for the increment C between $\kappa/12$ and $23\kappa/12$. Those κ's that do not meet the minimum requirements of the proof have been tested using a computer program. The algorithm is simply to test the parity and norm conditions for all possible ℓ and b values for each κ. The results were as expected, as those κ's which passed both conditions agreed with the known list of levels for which the modular invariants are non-trivial.

The third source of exceptionals was tested in the same manner. However, this source produces κ values of much greater magnitude than the previous ones.

Although these errant levels were successfully tested, one would hope that a better argument could be found that limits the number and size of the outlying κ's.

The exceptionals that were tested are:

- $6 \leq \kappa \leq 1498$ by (2.5) and §2.4.3
- $\kappa = \{2^i \cdot 3^j \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23\}$ by (2.14)

where $i \in \{1,2,3,4\}$, $j \in \{1,2\}$ and the parentheses indicate that any combination of prime or prime power factors may be chosen.

Chapter 3

Rank-Level Duality

In this chapter, we prove that the modular invariants for $C_{r,k}$ are exactly those of $C_{k,r}$. We do this by showing that the relationships between the S and T matrices of the two algebras preserve the parity and norm conditions. This *rank-level duality* is not restricted to the C family of affine Lie algebras. In fact, there exist similar dualities for all of the classical algebras. A duality exists in a slightly weaker form between $A_{n,k}$ and $A_{k,n}$, and between $so_k^{(1)}$ at level n and $so_n^{(1)}$ at level k. Together with the fact that $so_{3,k}$ is isomorphic to $C_{2,k}$, this implies that the $C_{2,k}$ classification of modular invariants is also the classification for $C_{k,2}$, $B_{n,3}$ and $D_{n,3}$, where B_n and D_n are the orthogonal algebras so_{2n+1} and so_{2n}, respectively. As we prove the rank-level duality of $C_{r,k}$, we will illustrate each step by explicitly working out the relevant quantities for $C_{2,k}$ and $C_{k,2}$.

3.1 Young diagrams

We must first establish a map between the primaries of $C_{r,k}$ and those of $C_{k,r}$. Let $\lambda = (\lambda_1, ..., \lambda_r)$ be a primary of $C_{r,k}$. This means that the λ_i are non-negative integers with $\lambda_1 + ... + \lambda_r \leq k$. The *Young diagram* of λ is defined to be the collection of boxes whose j^{th} row has length $\ell_j := \sum_{i=j}^{r} \lambda_i$, so that the size of the rows decreases as j increases. Note that for λ associated with $C_{r,k}$, the Young diagram for λ will have at most r rows (some of the λ_j's could be zero), with k being the maximum length of each row.

The transpose of a Young diagram (where the first row becomes the first column and so on) for $C_{r,k}$ will have at most k rows, with r being the maximum length of each row. This suggests that a possible map between the primaries of $C_{r,k}$ and those of $C_{k,r}$ is the transposition of the Young diagrams. Let λ^t denote a primary of $C_{k,r}$. Then, since transposition is an order two map, $(\lambda^t)^t$ will be equal to λ as long as it is interpreted as a primary for $C_{r,k}$. The Young diagram for λ associated to $C_{r,k}$ is unique, and so its transpose is also unique. The snare that arises is that if one is given a Young diagram without being told what r and k are, then there is

no way to determine these quantities. If the diagram has a rows, the first of which has length b, then all we know is that $r \geq a$ and $k \geq b$. However, the arrangement of boxes does uniquely determine the first a components of the primary in question, and the next $(r-a)$ of them must then be equal to zero. By ignoring these extra zeros, or, equivalently, by remembering r and k, we can consider transposition to be an invertible map. It is clearly onto, as the transpose of every Young diagram exists. Therefore $\lambda \mapsto \lambda^t$ is a bijective map.

We will now explicitly calculate λ^t in terms of λ. Write $\lambda^t = (\tilde{\lambda}_1, ..., \tilde{\lambda}_k)$. Let r_1 be the number of rows in the Young diagram of λ, so that

$$\sum_{i=1}^{k} \tilde{\lambda}_i = r_1 \leq r. \tag{3.1}$$

In the Young diagram for λ, there are exactly λ_i columns of length i. Thus, in the Young diagram for λ^t there are exactly λ_i rows of length i. Since we are interested in the individual $\tilde{\lambda}_j$'s, we need to know which of the λ_i's are nonzero. Let $R := \{r_1, r_2, ..., r_t\}$ be the descending set of subscripts of the nonzero components of λ. Then we can describe the Young diagram of λ^t: the first λ_{r_1} rows will have a length of r_1 boxes, the next λ_{r_2} rows will have length r_2, and the final λ_{r_t} rows will have r_t boxes. In other words,

$$\sum_{i=1}^{k} \tilde{\lambda}_i = ... = \sum_{i=\lambda_{r_1}}^{k} \tilde{\lambda}_i = r_1$$

$$\sum_{i=\lambda_{r_1}+1}^{k} \tilde{\lambda}_i = ... = \sum_{i=\lambda_{r_1}+\lambda_{r_2}}^{k} \tilde{\lambda}_i = r_2$$

$$\vdots$$

$$\sum_{i=k-\lambda_{r_t}+1}^{k} \tilde{\lambda}_i = ... = \sum_{i=k}^{k} \tilde{\lambda}_i = r_t.$$

The equations above allow us to conclude that

$$s_j = \sum_{i=1}^{j} \lambda_{r_i} \Rightarrow \tilde{\lambda}_{s_j} = r_j - r_{j+1}, \tag{3.2}$$

where j is between 1 and t and $r_{t+1} := 0$. All of the other $\tilde{\lambda}_i$'s are equal to zero. This is consistent with our initial assignment of r_1 in (3.1):

$$\begin{aligned} \sum_{i=1}^{k} \tilde{\lambda}_i &= \sum_{j=1}^{t} \tilde{\lambda}_{s_j} \\ &= (r_1 - r_2) + (r_2 - r_3) + ... + (r_{t-1} - r_t) + r_t \\ &= r_1. \end{aligned}$$

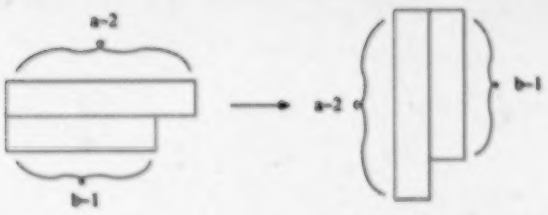

Figure 3.1: The Young diagrams of λ and λ^t for $C_{2,k}$.

Thus given any $\lambda \in P_+^k$ we can represent λ^t using the set R and equation (3.2).

For $\lambda \in P_+^k$ associated to $C_{2,k}$, $\lambda = (\lambda_1, \lambda_2)$ where $\lambda_1 + \lambda_2 \leq k$ and $\lambda_1, \lambda_2 \in \mathbb{Z}_{\geq}$. It is convenient to write $a := \lambda_1 + \lambda_2 + 2$ and $b := \lambda_2 + 1$. Then by definition, $\ell_1 = \lambda_1 + \lambda_2 = a - 2$ and $\ell_2 = \lambda_2 = b - 1$. The resulting Young diagram and its transpose are shown in Figure 3.1. We can ascertain the corresponding nonzero elements of λ^t just by looking at the picture: $\tilde{\lambda}_{b-1} = \tilde{\lambda}_{a-2} = 1$. This agrees perfectly with (3.2).

3.2 Jacobi's Theorem

Given an invertible $|L| \times |L|$ matrix Ω, define $(\Omega)_{IJ}$ to be the submatrix obtained from Ω by considering only those Ω_{ij} indexed by $i \in I$ and $j \in J$, where $I, J \subset L$. Then if $\bar{I} = L \setminus I$ and $\bar{J} = L \setminus J$, we have Jacobi's theorem [12]:

$$\det[((\Omega^{-1})^T)_{IJ}] = (-1)^{\sum_I + \sum_J} \cdot (\det \Omega)^{-1} \cdot \det[(\Omega)_{\bar{I}\bar{J}}], \tag{3.3}$$

where $\sum_I$ and $\sum_J$ are the sums of the elements of the sets I and J, respectively.

We are particularly interested in the following choice of Ω:

$$\Omega_{ij} := \sqrt{\frac{2}{\kappa}} \cdot \sin\left(\frac{\pi i j}{\kappa}\right), \quad \text{where } 1 \leq i, j < \kappa = r + k + 1. \tag{3.4}$$

In order to apply (3.3) we need to know $\det(\Omega)$ and $(\Omega^{-1})^T$. To this end, we interpret Ω as an S matrix for some $C_{1,k}$.

3.2.1 Ω as an S matrix

If we are looking at C_{1,k_1} for some level k_1, then for $\lambda, \mu \in P_+^{k_1}$, we have (as in (1.42)):

$$S_{\lambda\mu} = \sqrt{\frac{2}{k_1 + 2}} \cdot \sin\left(\pi \frac{\lambda[1]\,\mu[1]}{k_1 + 2}\right)$$

where $\lambda[1] = \lambda_1 + 1$ and $\mu[1] = \mu_1 + 1$. In order for this to agree with the entries of Ω given in (3.4), we must have $k_1 := \kappa - 2$. Then $\lambda, \mu \in P_+^{k_1}$ requires $0 \leq \lambda_1, \mu_1 \leq \kappa - 2$,

which implies that $1 \le \lambda_1 + 1 \le \kappa - 1$ and $1 \le \mu_1 + 1 \le \kappa - 1$. Therefore,

$$\begin{aligned}\Omega &= \sqrt{\frac{2}{\kappa}} \cdot \sin\left(\frac{\pi ij}{\kappa}\right)_{1 \le i,j \le \kappa-1} \\ &= (S_{\lambda\mu})_{\lambda,\mu \in P_+}.\end{aligned}$$

In other words, Ω is an S matrix. All S matrices are symmetric and unitary, so that $S = S^{-1}$, and by (3.4) Ω is symmetric. We have:

$$\Omega^{-1} = S^{-1} = S = \Omega \tag{3.5}$$

$$\Rightarrow \det(\Omega) = \pm 1 \tag{3.6}$$

Thus by (3.5) and (3.6) we know that $(\Omega^{-1})^T = \Omega$ and $\det(\Omega) = \pm 1$.

3.2.2 $S_{\lambda\mu}$ in terms of Ω

For $C_{r,k}$ primaries λ and μ, the matrix S can be written as follows [7]:

$$S_{\lambda\mu} = \left(\frac{2}{\kappa}\right)^{r/2} \cdot i^{r^2-r} \cdot \det\left(\sin\left(\pi \frac{\lambda[i]\,\mu[j]}{\kappa}\right)\right)_{1 \le i,j \le r} \tag{3.7}$$

where $\lambda[\ell] = r + 1 - \ell + \sum_{i=\ell}^{r} \lambda_i$. Note that $\lambda[\ell]$ decreases as ℓ increases, so that each $\lambda[\ell]$ is between 1 and $r + k = \kappa - 1$. An analogous definition and consequences hold for $\mu[\ell]$.

Let $I := \{\lambda[i]\}_{1 \le i \le r}$ and $J := \{\mu[j]\}_{1 \le j \le r}$. Then $(\Omega)_{IJ}$ is an r by r matrix and

$$\begin{aligned}\det[(\Omega)_{IJ}] &= \left(\sqrt{\frac{2}{\kappa}}\right)^{r} \cdot \det\left(\sin\left(\pi \frac{ij}{\kappa}\right)\right)_{i \in I, j \in J} \\ &= (-1)^{\frac{r(1-r)}{2}}\, S_{\lambda\mu}. \end{aligned} \tag{3.8}$$

Note that $r(1-r)$ is always even, so that $\det[(\Omega)_{IJ}] = \pm S_{\lambda\mu}$. When $r = 2$ we get a slight improvement, since now the sign is fixed.

3.2.3 $\tilde{S}_{\lambda^t\mu^t}$ in terms of Ω

We would like to find the analog of (3.8) for $\tilde{S}_{\lambda^t\mu^t}$. In order to apply (3.3), we need to write $\tilde{S}_{\lambda^t\mu^t}$ in terms of $(\Omega)_{IJ}$ for the sets I and J used in the previous section, namely $I := \{\lambda[i]\}_{1 \le i \le r}$ and $J := \{\mu[j]\}_{1 \le j \le r}$.

The $C_{k,r}$ analog of (3.7) is

$$\tilde{S}_{\lambda\mu} = \left(\frac{2}{\kappa}\right)^{k/2} \cdot i^{k^2-k} \cdot \det\left(\sin\left(\pi \frac{\tilde{\lambda}[i]\,\tilde{\mu}[j]}{\kappa}\right)\right)_{1 \le i,j \le k} \tag{3.9}$$

where $\tilde{\lambda}[\ell] = k + 1 - \ell + \sum_{i=\ell}^{k} \tilde{\lambda}_i$. Recall from §3.1 that $R := \{r_1, \ldots, r_t\}$ is the complete set of subscripts of the nonzero components of λ. This combined with

(3.2) gives us

$$\sum_{i=1}^{j-1}\lambda_{r_i} < \ell \le \sum_{i=1}^{j}\lambda_{r_i} \;\Rightarrow\; \tilde{\lambda}[\ell] = k+1-\ell+r_j, \tag{3.10}$$

while

$$r_{j+1} < \ell \le r_j \;\Rightarrow\; \lambda[\ell] = r+1-\ell+\sum_{i=1}^{j}\lambda_{r_i}. \tag{3.11}$$

We are interested in which integers are missing from I since those are the numbers that will be in $\bar{I}$. The following will prove very useful:

$$\ell \in (r_{j+1}, r_j] \;\Rightarrow\; \lambda[\ell] \in [r+1-r_j+\sum_{i+1}^{j}\lambda_{r_i}, r+1-r_{j+1}+\sum_{i=1}^{j}\lambda_{r_i})$$

$$\ell \in (r_j, r_{j-1}] \;\Rightarrow\; \lambda[\ell] \in [r+1-r_{j-1}+\sum_{i+1}^{j-1}\lambda_{r_i}, r+1-r_j+\sum_{i=1}^{j-1}\lambda_{r_i})$$

This means that as ℓ goes from r_j to r_j+1, $\lambda[\ell]$ skips over λ_{r_j} values, namely

$$r+1-r_j+\sum_{i=1}^{j-1}\lambda_{r_i}+x \tag{3.12}$$

where $x \in [0, \lambda_{r_j}-1]$. We would like to equate each of the values described in (3.12) with a $\tilde{\lambda}[\ell']$ for some ℓ' between 1 and k.

For ℓ as in (3.10), $\tilde{\lambda}[\ell]$ takes on exactly λ_{r_j} consecutive values, each of which looks like

$$k+1+r_j-\sum_{i=1}^{j-1}\lambda_{r_i}-(1+x) \tag{3.13}$$

for some $x \in [0, \lambda_{r_j}-1]$. There is a definite relationship between these values and those in (3.12). Fixing x and denoting the corresponding value given in (3.12) by y_1 and that of (3.13) by y_2, we have

$$y_1+y_2 = r+k+1 = \kappa.$$

This translates to $\bar{I} = \{\kappa - \tilde{\lambda}[\ell]\}_{1\le\ell\le k}$, rather than $\bar{I} = \{\tilde{\lambda}[\ell]\}_{1\le\ell\le k}$. In a similar manner, we find that $\bar{J} = \{\kappa - \tilde{\mu}[\ell]\}_{1\le\ell\le k}$.

In terms of λ^t associated to $C_{k,2}$ we see that

$$\lambda^t[\ell] = \begin{cases} 2 & \text{if } \ell \le b-1 \\ 1 & \text{if } b-1 < \ell \le a-2 \\ 0 & \text{if } \ell > a-2. \end{cases} \tag{3.14}$$

As ℓ goes from $(b-1)$ to b, $\lambda^t[\ell]$ decreases from $(k+4-b) = (\kappa+1-b)$ to $(\kappa-1-b)$, which implies that $\lambda^t[\ell] \neq \kappa - b$ for any ℓ between 1 and k. Similarly, $\lambda^t[\ell] \neq \kappa - a$. Note that $\lambda[1] = a$ and $\lambda[2] = b$, so that we have $I = \{a, b\}$ and $\bar{I} = \{\kappa - a, \kappa - b\}$.

Let $\mathcal{J}$ be a simple current for $C_{k,r}$. By [7], $\mathcal{J}$ has order 2 and acts on $\lambda^t \in P_+^r$ by reversing the order of the $\tilde{\lambda}_i$'s (where $\tilde{\lambda}_i$ is written as an $(r+1)$-tuple, i.e. we include the 0^{th} node). This means that $\mathcal{J}(\{\tilde{\mu}[\ell]\}_{1\le\ell\le k}) = \{\kappa - \tilde{\mu}[\ell]_{1\le\ell\le k}\}$. Then by (1.15) and (1.17), we have:

$$\begin{aligned}\det[(\Omega)_{IJ}] &= (-1)^{\frac{k(1-k)}{2}} S_{\mathcal{J}\lambda^t,\mathcal{J}\mu^t} \\ &= \exp[2\pi i\, Q_{\mathcal{J}}(\mathcal{J}\lambda^t)]\, \exp[2\pi i\, Q_{\mathcal{J}}(\mu^t)]\, S_{\lambda^t\mu^t}, \end{aligned} \tag{3.15}$$

where $Q_{\mathcal{J}}(\lambda^t) = \sum_{j=1}^{k} j\tilde{\lambda}_j/2$. Note that $Q_{\mathcal{J}}(\mathcal{J}\lambda^t) = Q_{\mathcal{J}}(\mathcal{J}0) - Q_{\mathcal{J}}(\lambda^t)$ and $Q_{\mathcal{J}}(\mathcal{J}0) = kr/2$. From the definitions of I, J and Q we get

$$(-1)^{\Sigma_I+\Sigma_J} = (-1)^{2(Q_{\mathcal{J}}(\lambda)+Q_{\mathcal{J}}(\mu))}. \tag{3.16}$$

Our final observation is that for any primary λ, the Young diagram of λ contains exactly $2Q$ boxes. This means that $Q(\lambda) = Q(\lambda^t)$.

For $C_{2,k}$, the simple current action on $\lambda = (k-(\lambda_1+\lambda_2);\lambda_1,\lambda_2)$ is $\mathcal{J}(\lambda) = (\lambda_2;\lambda_1,k-(\lambda_1+\lambda_2)) = (b-1;a-(b+1),\kappa-(a+1))$. Then $\mathcal{J}(\lambda)[1] = 2+(a-(b+1))+(\kappa-(a+1)) = (\kappa-b)$ and $\mathcal{J}(\lambda)[2] = 1+(\kappa-(a+1)) = (\kappa-a)$, as expected.

3.2.4 Conclusion

We now combine all of our results to show that $S_{\lambda\mu} = \tilde{S}_{\lambda^t\mu^t}$. By Jacobi's theorem (3.3), and by (3.8) and (3.15), we have

$$\begin{aligned}(-1)^{\frac{r(1-r)}{2}} S_{\lambda\mu} &= (-1)^{\Sigma_I+\Sigma_J}(\pm 1)\cdot\exp[2\pi i(Q_{\mathcal{J}}(\mu) - Q_{\mathcal{J}}(\lambda^t) + kr/2)]\,\tilde{S}_{\lambda^t\mu^t} \\ &= (-1)^{2(Q_{\mathcal{J}}(\lambda)+Q_{\mathcal{J}}(\mu))}\,(\pm 1)\,(-1)^{2(Q_{\mathcal{J}}(\mu)-Q_{\mathcal{J}}(\lambda^t)+kr/2)}\,\tilde{S}_{\lambda^t\mu^t} \\ \Rightarrow S_{\lambda\mu} &= (\pm 1)\cdot(-1)^{kr+\frac{1}{2}(r^2-r+k-k^2)}\,\tilde{S}_{\lambda^t\mu^t}.\end{aligned}$$

Thus, $S_{\lambda\mu} = \pm\tilde{S}_{\lambda^t\mu^t}$ where the sign does not depend on either λ or μ. Both S and $\tilde{S}$ are modular data matrices, so they must obey (1.6b), i.e. the entries in their first row and column must be positive. This forces the sign to be positive, and we conclude that

$$S_{\lambda\mu} = \tilde{S}_{\lambda^t\mu^t} \tag{3.17}$$

for all λ, $\mu \in P_+^k$ and λ^t, $\mu^t \in P_+^r$.

3.3 T matrix duality

We now consider the T matrix associated to $C_{r,k}$. Recall from (1.38) that

$$T_{\lambda\lambda} = \gamma \exp\left[\frac{\pi i(\lambda+\rho|\lambda+\rho)}{\kappa}\right]$$

where γ is a constant that does not depend on λ. We are interested in the dot products $(\lambda+\rho)^2$ and $(\lambda^t+\tilde{\rho})^2$ for any $\lambda \in P_+^k$, where λ^t is defined in terms of the

transpose of the Young diagram of λ (as in §3.1). Our first result is that for any simple current $\mathcal{J}$ of $C_{k,r}$, we have the following:

$$(\lambda+\rho)^2+(\mathcal{J}\lambda^t+\tilde{\rho})^2=\frac{1}{2}\sum_{i=1}^{\kappa-1}i^2=\frac{\kappa(\kappa-1)(2\kappa-1)}{12}. \tag{3.18}$$

Proof. First note that relative to the orthogonal basis given in [2], the ℓ^{th} component of $(\lambda+\rho)$ looks like

$$\lambda_\ell+...+\lambda_r+r+(1-\ell)=r+1-\ell+\sum_{i=\ell}^{r}=\lambda[\ell]. \tag{3.19}$$

As we saw in §3.2.3, the simple current $\mathcal{J}$ acts on the set of $\tilde{\lambda}[\ell]$'s by $\{\mathcal{J}\tilde{\lambda}[\ell]\}=\{\kappa-\tilde{\lambda}[\ell]\}$, and the sum of the $\mathcal{J}[\ell]$'s with the $\lambda[\ell]$'s is equal to the sum of all the numbers from 1 to $r+k=\kappa-1$. From these observations we get (3.19). □

Equation (3.18) holds for all $\lambda\in P_+^k$, and in particular for $\lambda=0$. We have

$$T_{\lambda\lambda}\tilde{T}_{\mathcal{J}\lambda^t,\mathcal{J}\lambda^t}=T_{00}\tilde{T}_{\mathcal{J}\tilde{0},\mathcal{J}\tilde{0}}, \tag{3.20}$$

where $\tilde{T}$ is the matrix corresponding to λ^t and $\tilde{0}$ is the vacuum for $C_{k,r}$. By [8], the simple current symmetry of the T matrix is given by

$$T_{\mathcal{J}a,\mathcal{J}a}\overline{T}_{aa}=\overline{\varphi_{\mathcal{J}}(a)}\,T_{\mathcal{J}0,\mathcal{J}0}\overline{T}_{00}, \tag{3.21}$$

and so we have

$$\tilde{T}_{\mathcal{J}\lambda^t,\mathcal{J}\lambda^t}\overline{\tilde{T}}_{\lambda^t\lambda^t}=\overline{\varphi_{\mathcal{J}}(\lambda^t)}\,\tilde{T}_{\mathcal{J}\tilde{0},\mathcal{J}\tilde{0}}\overline{\tilde{T}}_{\tilde{0}\tilde{0}}. \tag{3.22}$$

By (1.6a) $T_{\lambda\lambda}$ is a root of unity, and so (3.20) and (3.22) tell us that

$$T_{\lambda\lambda}\tilde{T}_{\lambda^t\lambda^t}=\varphi_{\mathcal{J}}(\lambda)T_{00}\tilde{T}_{\tilde{0}\tilde{0}}. \tag{3.23}$$

We would like to show that $MT=TM$ if and only if $\tilde{M}\tilde{T}=\tilde{T}\tilde{M}$, so that $\tilde{M}$ satisfies the modular invariant property (1.12a) whenever M does. First suppose that M is a modular invariant for $C_{r,k}$ and recall from (1.13) that $M_{\lambda\mu}\neq 0\Rightarrow T_{\lambda\lambda}=T_{\mu\mu}$. We prove the following selection rule for $C_{r,k}$ (which should not be confused with (1.19), where the assumption $M_{\mathcal{J}0,\mathcal{J}0}\neq 0$ is required):

$$M_{\lambda\mu}\neq 0\Rightarrow\varphi_{\mathcal{J}}(\lambda)=\varphi_{\mathcal{J}}(\mu). \tag{3.24}$$

Proof. For $C_{r,k}$, by (1.17), we know that $\varphi_{\mathcal{J}}(\lambda)=\varphi_{\mathcal{J}}(\mu)$ iff $\sum_{i=1}^r i\lambda_i\equiv\sum_{j=1}^r j\mu_j$ (mod 2). By (1.13), we have $T_{\lambda\lambda}=T_{\mu\mu}$, and given the definition of T this implies that $(\lambda+\rho)^2\equiv(\mu+\rho)^2\pmod{2\kappa}$. Hence,

$$\sum_{i=1}^{r}\left(\sum_{j=i}^{r}\lambda_i+(r-i+1)\right)\equiv\sum_{i=1}^{r}\left(\sum_{j=i}^{r}\mu_i+(r-i+1)\right)\pmod 2$$

$$\Rightarrow\qquad\sum_{i=1}^{r}i\lambda_i+\frac{r(r+1)}{2}\equiv\sum_{i=1}^{r}i\mu_i+\frac{r(r+1)}{2}\pmod 2$$

and we have $\varphi_{\mathcal{J}}(\lambda)=\varphi_{\mathcal{J}}(\mu)$ as required. □

Now by (3.23) we can conclude that $\tilde{T}_{\lambda^t\lambda^t} = \tilde{T}_{\mu^t\mu^t}$ for all λ^t, μ^t, and this is equivalent to $\tilde{M}\tilde{T} = \tilde{T}\tilde{M}$. The opposite implication is proved in the same way.

Based on the results above and those of §3.2.4, we may now conclude that M is a modular invariant for $C_{2,k}$ if and only if the corresponding matrix $\tilde{M}$ is a modular invariant for $C_{k,r}$.

Chapter 4

Automorphism Modular Invariants

The automorphism modular invariants of all the classical affine Lie algebras were classified in [11]. This chapter specialises their work to the affine algebra $C_{2,k}$.

Whenever a modular invariant M is a permutation matrix, we refer to M as an *automorphism invariant*, and associate to it the permutation σ of P_+^k that satisfies $M_{\lambda\mu} = \delta_{\mu,\sigma(\lambda)}$. This definition of σ is equivalent to requiring that

$$S_{\lambda\mu} = S_{\sigma(\lambda)\sigma(\mu)}, \quad \text{and} \tag{4.1}$$

$$T_{\lambda\lambda} = S_{\sigma(\lambda)\sigma(\lambda)} \tag{4.2}$$

for the matrices S and T defined in (1.36). We will show that for $C_{2,k}$ there are at most two distinct automorphism invariants.

We first establish that the sets of weights of $C_{2,k}$ which have the second and third smallest quantum-dimensions (see (4.3) below) are necessarily fixed by any automorphism invariant permutation σ. Then we determine these sets explicitly and use them to narrow down the possibilities for σ. The final step is to use the norm condition to fully determine the automorphism invariants.

4.1 Preliminary results

Using (1.39) and the denominator identity mentioned in §1.6.1, we can write the quantum-dimensions $S_{0\lambda}/S_{00}$ in terms of the Weyl denominator formula as follows:

$$\mathcal{D}(\lambda) := \frac{S_{0\lambda}}{S_{00}} = \prod_{\alpha>0} \frac{\sin(\pi\alpha\cdot(\rho+\lambda)/\kappa)}{\sin(\pi\alpha\cdot\rho/\kappa)}. \tag{4.3}$$

Let $[\lambda] := \{\lambda, \mathcal{J}\lambda\}$ where $\mathcal{J}$ is the simple current involution given by $\mathcal{J} : (\lambda_0, \lambda_1, \lambda_2) \mapsto (\lambda_2, \lambda_1, \lambda_0)$. Note that the simple current property (1.15) combined with (1.17) implies that $\mathcal{D}(\lambda)$ is constant on $[\lambda]$:

$$S_{\mathcal{J}\lambda,0} = \varphi_{\mathcal{J}}(0)S_{\lambda 0} = \exp[2\pi i Q_{\mathcal{J}}(0)]S_{\lambda 0} = S_{\lambda 0}.$$

By [6] we know that $\mathcal{D}(\lambda) = 1$ if and only if $\lambda \in [0] = \{0, \mathcal{J}0\}$; in §1.4 we found that $S_{0\lambda} \geq S_{00}$ for all λ. This means that $\mathcal{D}(\lambda)$ achieves its minimum value on $[0]$. Let $\mathcal{E}_2$ and $\mathcal{E}_3$ represent the set of weights on which $\mathcal{D}(\lambda)$ attains its second and third smallest value, respectively, and let $\lambda' \in \mathcal{E}_m$ for $m = 2, 3$. Then by (4.1) and the fact that σ must fix zero (otherwise $S_{\lambda 0} \geq 0$ will be contradicted), $\mathcal{D}(\sigma\lambda') = D(\lambda')$. This means that $\sigma\lambda'$ has the m^{th} smallest q-dimension if and only if λ' does, and so

$$\sigma\mathcal{E}_m = \mathcal{E}_m \qquad \text{for } m = 2, 3. \tag{4.4}$$

Equation (4.4) provides an additional constraint on our automorphism invariants. We will find that whenever k is greater than three, $\mathcal{E}_2 = [\omega^1]$ for the fundamental weight ω^1. Thus (4.4) implies that either $\sigma(\omega^1) = \mathcal{J}(\omega^1)$ or $\sigma(\omega^1) = \omega^1$. When k is odd we see that both actions of σ are possible, while only the latter occurs for even k.

4.2 Candidates for $\mathcal{E}_2$

Our task in this section is to prove the following lemma. In §4.3 we will precisely determine $\mathcal{E}_2$ for each level k.

Lemma 3. *The candidates for $\mathcal{E}_2$ are $[\omega^1]$, $[\omega^2]$ and $[k\omega^1]$.*

Let a and $b \neq 0$ be vectors in $\mathbb{R}^3$ and suppose that for all $t \in [t_0, t_1]$, $a + bt \in \overline{P}_+^k$, where $\overline{P}_+^k$ is the same as the usual P_+^k without the requirement that all the components be integers. Using the definition (4.3), we show that $\mathcal{D}(a+bt)$ achieves its minimum value at one of t_0, t_1. To simplify the following terms, let $\beta := (\rho + a + bt)/\kappa$. We denote the four positive roots of C_2 by $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$ [2]. Now,

$$\begin{aligned}
\mathcal{D}(a + bt) &= \prod_{\alpha > 0} \frac{\sin(\pi\alpha \cdot \beta)}{\sin(\pi\alpha \cdot \rho/\kappa)} \\
\Rightarrow \frac{d}{dt}\mathcal{D}(a + bt) &= \Big[\frac{\pi}{\kappa}(\alpha_1 \cdot b)\cos[\pi\alpha_1 \cdot \beta] \cdot \sin[\pi\alpha_2 \cdot \beta] \\
&\quad \cdot \sin[\pi(\alpha_1 + \alpha_2) \cdot \beta] \cdot \sin[\pi(2\alpha_1 + \alpha_2) \cdot \beta] \;+\; \text{similar terms}\Big] \\
&\quad \prod_{\alpha > 0} \frac{1}{\sin(\pi\alpha \cdot \rho/\kappa)}.
\end{aligned}$$

The similar terms alluded to in the formula above are the three products involving the cosine of each of α_2, $\alpha_1 + \alpha_2$ and $2\alpha_1 + \alpha_2$ that are analogous to the one given.

As a function of t, $\mathcal{D}$ will take on its extreme value on the interval $[t_0, t_1]$ at some

point t' such that $\mathcal{D}(a+bt')=0$. We have:

$$\frac{d^2}{dt^2}\mathcal{D}(a+bt) = \frac{\pi}{\kappa}\cdot\Big[\frac{\pi}{\kappa}(\alpha_1\cdot b)^2(-\sin(\pi\alpha_1\cdot\beta))\cdot\sin(\pi\alpha_2\cdot\beta)\cdot\sin(\pi(\alpha_1+\alpha_2)\cdot\beta)\cdot$$
$$\sin(\pi(2\alpha_1+\alpha_2)\cdot\beta) \;+\; \frac{\pi}{\kappa}(\alpha_1\cdot b)(\alpha_2\cdot b)\cos(\pi\alpha_1\cdot\beta)\cdot\cos(\pi\alpha_2\cdot\beta)\cdot$$
$$\sin(\pi(\alpha_1+\alpha_2)\cdot\beta)\sin(\pi(2\alpha_1+\alpha_2)\cdot\beta) \;+\; \frac{\pi}{\kappa}(\alpha_1\cdot b)((\alpha_1+\alpha_2)\cdot b)$$
$$\cos(\pi\alpha_1\cdot\beta)\cdot\sin(\pi\alpha_2\cdot\beta)\cdot\cos(\pi(\alpha_1+\alpha_2)\cdot\beta)\cdot\sin(\pi(2\alpha_1+\alpha_2)\cdot\beta) +$$
$$\frac{\pi}{\kappa}(\alpha_1\cdot b)((2\alpha_1+\alpha_2)\cdot b)\cos(\pi\alpha_1\cdot\beta)\cdot\sin(\pi\alpha_2\cdot\beta)\cdot\sin(\pi(\alpha_1+\alpha_2)\cdot\beta)\cdot$$
$$\cos(\pi(2\alpha_1+\alpha_2)\cdot\beta) \;+\; \text{similar terms}\Big]\cdot\prod_{\alpha>0}\frac{1}{\sin(\pi\alpha\cdot\rho/\kappa)}.$$

We are assuming that $\frac{d}{dt}\mathcal{D}(a+bt')=0$. In order to use this information we need to rewrite the sums and products found in $\frac{d^2}{dt^2}\mathcal{D}$. Ignoring the constants (i.e. the expressions that are independent of t), the terms that have been explicitly calculated above simplify to:

$$\frac{\pi}{\kappa}(\alpha_1\cdot b)\,\frac{\cos(\pi\alpha_1\cdot\beta)}{\sin(\pi\alpha_1\cdot\beta)}\Big[\mathcal{D}'(a+bt) - \Big(\cos(\pi\alpha_1\cdot\beta)^{-1}\cdot\sin(\pi\alpha_2\cdot\beta)\cdot$$
$$\sin(\pi(\alpha_1+\alpha_2)\cdot\beta)\cdot\sin(\pi(2\alpha_1+\alpha_2)\cdot\beta)\Big)\Big].$$

Now letting $t=t'$ and including all of the terms in $\mathcal{D}'$, we get

$$\mathcal{D}''(a+bt) = -\frac{\pi^2}{\kappa^2}\cdot\prod_{\alpha>0}\frac{\sin(\pi\alpha\cdot\beta/\kappa)}{\sin(\pi\alpha\cdot\rho/\kappa)}\left[\sum_{\alpha>0}\frac{(\alpha+b)^2}{\sin^2(\pi\alpha\cdot\beta/\kappa)}\right]$$
$$= -\mathcal{D}(a+bt)\cdot\frac{\pi^2}{\kappa^2}\cdot\sum_{\alpha>0}\frac{(\alpha+b)^2}{\sin^2(\pi\alpha\cdot\beta/\kappa)}. \tag{4.5}$$

By definition, $\mathcal{D}>0$, and all the other terms in (4.5) are squared, so we may conclude that if $\mathcal{D}'(a+bt)=0$, then $\mathcal{D}''(a+bt)<0$. In other words, $\mathcal{D}$ attains a local maximum at $t=t'$, and so its minimum value occurs at one of the endpoints $t_0,\ t_1$. Thus,

$$\mathcal{D}(a+bt') \geq \min\{\mathcal{D}(a+bt_0),\ \mathcal{D}(a+bt_1)\} \text{ whenever } t'\in[t_0,t_1]. \tag{4.6}$$

Lemma 4. *Suppose that $m=(m_1,m_2,m_3)\in\mathbb{Z}^3$, with $m_0+m_1+m_2=0$, and not all $m_i=0$. Then if $\lambda\in\mathcal{E}_2$ and $\lambda\pm m\notin[0]$, $\lambda_i<|m_i|$ for some $0\leq i\leq 2$.*

Proof. Let $\lambda\in P_+^k$ and consider $\lambda(t)=\lambda+mt$. Note that $\lambda(t)\in\overline{P}_+^k$, with $\lambda(t)\in P_+^k$ if t is an integer. Define

$$t_0 = \max_{i:m_i>0}\left(\frac{-\lambda_i}{m_i}\right) \quad\text{and}\quad t_1 = \min_{i:m_i<0}\left(\frac{-\lambda_i}{m_i}\right). \tag{4.7}$$

Now, if $t_0<-1$ and $t_1>1$, then both $\lambda(1)=\lambda+m$ and $\lambda(-1)=\lambda-m$ must be in P_+^k. From (4.6),

$$\mathcal{D}(\lambda+mt) \geq \min\{\mathcal{D}(\lambda+m),\ \mathcal{D}(\lambda-m)\}$$

for all t between -1 and 1. In particular, $\mathcal{D}(\lambda(0)) = \mathcal{D}(\lambda) \geq \min\{\mathcal{D}(\lambda + m), \mathcal{D}(\lambda - m)\}$. This contradicts our hypothesis that $\lambda \pm m \notin [0]$ while $\lambda \in \mathcal{E}_2$. Thus either $-1 < t_0$ or $t_1 < 1$. Given our definitions in (4.7), this is equivalent to the existence of an i such that $\lambda_i < |m_i|$. □

We can use Lemma 4 to get a short list of possible candidates for $\mathcal{E}_2$. Consider $m = (1, -1, 0)$, $m' = (1, 0, -1)$ and $m'' = (0, -1, 1)$. We begin with the assumption that $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathcal{E}_2$, while $\lambda \pm \{m, m', m''\} \notin [0]$. Our first m tells us that either $\lambda_0 = 0$ or $\lambda_1 = 0$ (recall that all the λ_i's are positive integers). The next vector, m', implies that one of λ_0 and λ_2 is zero. Note that if both λ_1 and λ_2 are zero, then $\lambda = 0$, which is not in $\mathcal{E}_2$. Therefore $\lambda_0 = 0$ and so none of $\lambda \pm \{m, m', m''\}$ are in $[0] = \{(k, 0, 0), (0, 0, k)\}$. Our final vector m'' forces either λ_1 or λ_2 to be zero. If λ_2 is the nonzero component then $\lambda = (0, 0, k) = \mathcal{J}(0)$ which is not in $\mathcal{E}_2$. However, $\lambda = (0, k, 0) = k\omega^1$ is a possibility.

While it may seem as though we have found only one candidate for $\mathcal{E}_2$, there are others, namely $[0] \pm \{m, m', m''\}$. By checking each of these twelve vectors, we find that only four of them are in P_+^k. They are: $\{(0 - m) = \omega^1, (0 - m') = \omega^2, (\mathcal{J}(0) - m'') = \mathcal{J}(\omega^1), (\mathcal{J}(0) + m') = \mathcal{J}(\omega^2)\}$. We have proved Lemma 3.

4.3 Determining $\mathcal{E}_2$

Our main result is that $\mathcal{E}_2 = [\omega^1]$ unless $k = 2$ or 3. If $k = 2$ then $\mathcal{E}_2 = [\omega^2] \cup [2\omega^1]$ and $\mathcal{E}_3 = [\omega^1]$, while $k = 3$ implies that $\mathcal{E}_2 = [\omega^1] \cup [\omega^2] \cup [3\omega^1]$.

We first look at those candidates for $\mathcal{E}_2$ that are not dependent on k (except in the 0^{th} node). By Lemma 3, these are $[\omega^1]$ and $[\omega^2]$. We need to determine the relationship between $\mathcal{D}(\omega^1)$ and $\mathcal{D}(\omega^2)$, and whichever is smaller will then be compared to $\mathcal{D}(k\omega^1)$. The distinction between those candidates that are dependent on k and those that are not is made so that we can differentiate $\mathcal{D}_k(\lambda)$ with respect to k.

Using (4.3) we find that

$$\frac{\mathcal{D}_k(\omega^2)}{\mathcal{D}_k(\omega^1)} = \prod_{\alpha>0} \frac{\sin[\pi\alpha \cdot (\rho + \omega^2)/\kappa]}{\sin[\pi\alpha \cdot (\rho + \omega^1)/\kappa]} = \frac{\sin(\frac{5\pi}{2\kappa})}{2\sin(\frac{\pi}{\kappa})\cos(\frac{\pi}{2\kappa})}. \tag{4.8}$$

The dot products $(\omega^i \cdot \alpha_j)$ are computed relative to the orthogonal coordinates $\{e_i \mid e_i \cdot e_j = \frac{1}{2}\delta_{i,j}\}$ (as in [2]). For C_2, $\omega^1 = \frac{1}{\sqrt{2}}(1, 0)$, $\omega^2 = \frac{1}{\sqrt{2}}(1, 1)$, $\alpha_1 = \frac{1}{\sqrt{2}}(1, -1)$ and $\alpha_2 = \frac{1}{\sqrt{2}}(0, 2)$.

We can show that (4.8) is increasing for all $k > 3$ by taking its derivative. We find that $\frac{\partial}{\partial k}\frac{\mathcal{D}_k(\omega^2)}{\mathcal{D}_k(\omega^1)} > 0$ for all $k > 0$ (i.e. for $\kappa \geq 4$). Now if there exists a k_0 such that $\mathcal{D}_{k_0}(\omega^2) \geq \mathcal{D}_{k_0}(\omega^1)$, then (4.8) tells us that $\mathcal{D}_k(\omega^2)$ will be strictly greater than

$\mathcal{D}_k(\omega^1)$ for all $k > k_0$. By testing k values in (4.8) we find that $k_0 = 3$:

$$\frac{\mathcal{D}_3(\omega^2)}{\mathcal{D}_3(\omega^1)} = \frac{\sin(\frac{5\pi}{12})}{2\sin(\frac{\pi}{6})\cos(\frac{\pi}{12})} = 1. \tag{4.9}$$

This means that ω^2 will no longer be a candidate for $\mathcal{E}_2$ whenever $k > 3$. If $k = 2$ or 3 we calculate the ratios of the q-dimensions directly. When $k = 1$ note that $[\omega^2] = [0]$, and so $\mathcal{E}_2 = [\omega^1]$ automatically.

Now we consider $[k\omega^1]$ (note that $k\omega^1$ is fixed by $\mathcal{J}$). The rank-level duality properties of $C_{2,k}$ as detailed in the previous chapter greatly simplify our task. Note that by (3.17)

$$\tilde{\mathcal{D}}(\lambda^t) = \frac{\tilde{S}_{\lambda' 0}}{\tilde{S}_{00}} = \frac{S_{\lambda 0}}{S_{00}} = \mathcal{D}(\lambda) \tag{4.10}$$

and so we must have $\lambda \in \mathcal{E}_2$ if and only if $\lambda^t \in \tilde{\mathcal{E}}_2$, where tilde's are used to represent quantities in $C_{k,2}$. Furthermore, we can calculate $(\omega^1)^t$ and $(k\omega^1)^t$ directly:

$$(\omega^i)^t = i\tilde{\omega}^1 \quad \text{and} \quad (j\omega^i)^t = i\tilde{\omega}^j. \tag{4.11}$$

Thus by (4.9) and (4.11),

$$\mathcal{D}(k\omega^1) = \tilde{\mathcal{D}}(\tilde{\omega}^k) > \tilde{\mathcal{D}}(\tilde{\omega}^1) = \mathcal{D}(\omega^1), \tag{4.12}$$

and we conclude that $\mathcal{E}_2 = [\omega^1]$ for all $k > 3$.

If $k = 2$, the candidates for $\mathcal{E}_2$ are $[\omega^1]$, $[\omega^2]$ and $[2\omega^1]$. By direct calculation we find that $\mathcal{D}_2(\omega^2) < \mathcal{D}_2(\omega^1)$ and $\mathcal{D}_2(\omega^2) = \mathcal{D}(2\omega^1)$:

$$\frac{\mathcal{D}_2(\omega^2)}{\mathcal{D}_2(\omega^1)} = \frac{\sin(\frac{5\pi}{10})}{2\sin(\frac{\pi}{5})\cos(\frac{\pi}{10})} < 1$$

and,

$$\frac{\mathcal{D}_2(\omega^2)}{\mathcal{D}_2(2\omega^1)} = \frac{\sin(\frac{\pi}{10})\sin(\frac{2\pi}{5})\sin(\frac{3\pi}{5})}{\sin(\frac{3\pi}{10})\sin(\frac{\pi}{5})\sin(\frac{4\pi}{5})} = 1.$$

This implies that $\mathcal{E}_2 = [\omega^2] \cup [2\omega^1]$. For $C_{2,2}$, $P_+^k = \{0, \mathcal{J}0, \omega^1, \mathcal{J}\omega^1, \omega^2, 2\omega^1\}$ so by the process of elimination we must have $\mathcal{E}_3 = [\omega^1]$.

When $k = 3$ we have $\mathcal{D}_3(\omega^1) = \mathcal{D}_3(\omega^2) = \mathcal{D}_3(3\omega^1)$:

$$\frac{\mathcal{D}_3(3\omega^1)}{\mathcal{D}_3(\omega^1)} = \frac{\sin(\frac{\pi}{6})\sin(\frac{5\pi}{6})}{\sin(\frac{\pi}{12})\sin(\frac{5\pi}{12})} = 1.$$

Thus $\mathcal{E}_2 = [\omega^1] \cup [\omega^2] \cup [3\omega^1]$ for $k = 3$, and we have justified the results stated at the beginning of this section. The next step is to actually find σ.

4.4 Classifying the Automorphism Invariants

We are now ready to determine all of the possible automorphism invariants for $C_{2,k}$. The results of the previous section, combined with (4.4), imply that any automorphism invariant permutation σ must fix $[\omega^1]$. This means that $\sigma(\omega^1) = \mathcal{J}^a(\omega^1)$ for some $a \in \{0,1\}$. If $a = 1$ then we apply the norm condition to get

$$(\rho + \omega^1)^2 \equiv (\rho + \mathcal{J}\omega^1)^2 \pmod{2\kappa} \Rightarrow (k-1)\kappa \equiv 0 \pmod{2\kappa}, \tag{4.13}$$

so that k must be odd. It happens that by [1], another automorphism invariant exists under exactly the same condition: If k is odd then the permutation $\sigma_{\mathcal{J}}$ given by

$$\sigma_{\mathcal{J}}(\lambda) = \mathcal{J}^{\lambda_1}(\lambda) \tag{4.14}$$

defines an automorphism invariant. Note that $\sigma_{\mathcal{J}}(\omega^1) = \mathcal{J}^1(\omega^1) = \sigma(\omega^1)$. This means that by replacing σ with $\sigma_{\mathcal{J}}^{-1}$ composed with σ, we find that our new σ fixes ω^1 for all k.

If $k = 1$, then $P_+^k = \{(0,0),(1,0),(0,1)\} = \{0,\omega^1,\omega^2\}$. Then the simple current action is

$$\sigma_{\mathcal{J}}(\lambda) = \begin{cases} \mathcal{J}(\lambda) & \text{if } \lambda = \omega^1 \\ \lambda & \text{if } \lambda \neq \omega^1 \end{cases} \tag{4.15}$$

and so $\sigma_{\mathcal{J}} = \sigma_1$, the trivial permutation.

The next step is to use the fusion product of ω^1 with itself to show that, by the norm condition, σ must fix ω^2 as well. Then by Lemma 5 below, we can conclude that σ is the identity permutation.

Recall from §1.5 that if λ and ρ are elements of the fusion ring, then $\lambda \times \mu = \sum N_{\lambda\mu}^{\nu}\nu$. We can use (1.40) to calculate the fusion product of certain weights. By [11], we have the fusion product

$$\omega^1 \times \omega^1 = \omega^2 + 2\omega^1. \tag{4.16}$$

The norm of the left hand side of (4.16) is fixed by σ, and so this must also be the case for the right side. Let $\sigma(\omega^2) = (c,d)$ for some nonnegative integers c and d. Then $(\rho + \omega^2 + 2\omega^1)^2 = (\rho + \sigma(\omega^2) + 2\omega^1)^2$ implies that $c^2 + d^2 + 8c + 2d = 12$. The only possible solution is that $c = d = 1$, which implies that σ fixes ω^2, and so by Lemma 5 below we may conclude that $\sigma = \mathrm{id}$. Thus the only non-trivial automorphism invariant permutation is $\sigma_{\mathcal{J}}$, which occurs when k is odd.

Lemma 5. *If σ fixes ω^1 and ω^2, then σ is trivial on P_+^k.*

Proof. By [9] we know that $S_{\lambda\mu}/S_{0\mu}$ can be written as a polynomial P_λ in $S_{\omega^i\mu}/S_{0\mu}$.

Applying σ to P_λ we get:

$$\begin{aligned} P_{\sigma(\lambda)} &= \frac{S_{\sigma(\lambda)\sigma(\mu)}}{S_{\sigma(0)\sigma(\mu)}} \\ &= \frac{S_{\lambda\mu}}{S_{0\mu}} \qquad \text{by (4.1)} \\ &= P_\lambda. \end{aligned}$$

Now, P_λ depends on $S_{\omega^i\mu}/S_{0\mu}$, and so this ratio must be fixed under σ. By our hypothesis the ω^i are fixed by σ, and we always have $\sigma(0) = 0$. Therefore,

$$\begin{aligned} \frac{S_{\omega^i\mu}}{S_{0\mu}} &= \frac{S_{\sigma(\omega^i)\sigma(\mu)}}{S_{\sigma(0)\sigma(\mu)}} \\ &= \frac{S_{\omega^i\sigma(\mu)}}{S_{0\sigma(\mu)}} \end{aligned}$$

and we may conclude that $S_{\lambda\mu} = S_{\lambda\sigma(\mu)}$ for all λ, μ. This is a contradiction to the fact that S is an invertible matrix unless σ is the identity mapping. □

We conclude this chapter by remarking upon the significance of the classification of automorphism invariants. We have shown that for the affine algebra $C_{2,k}$, the only nontrivial automorphism invariant acts like a simple current and exists when k is odd. This suggests that given Conjecture 1, §1.6.5, for odd k, our modular invariant classification exactly matches the list of known modular invariants contained in that section.

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